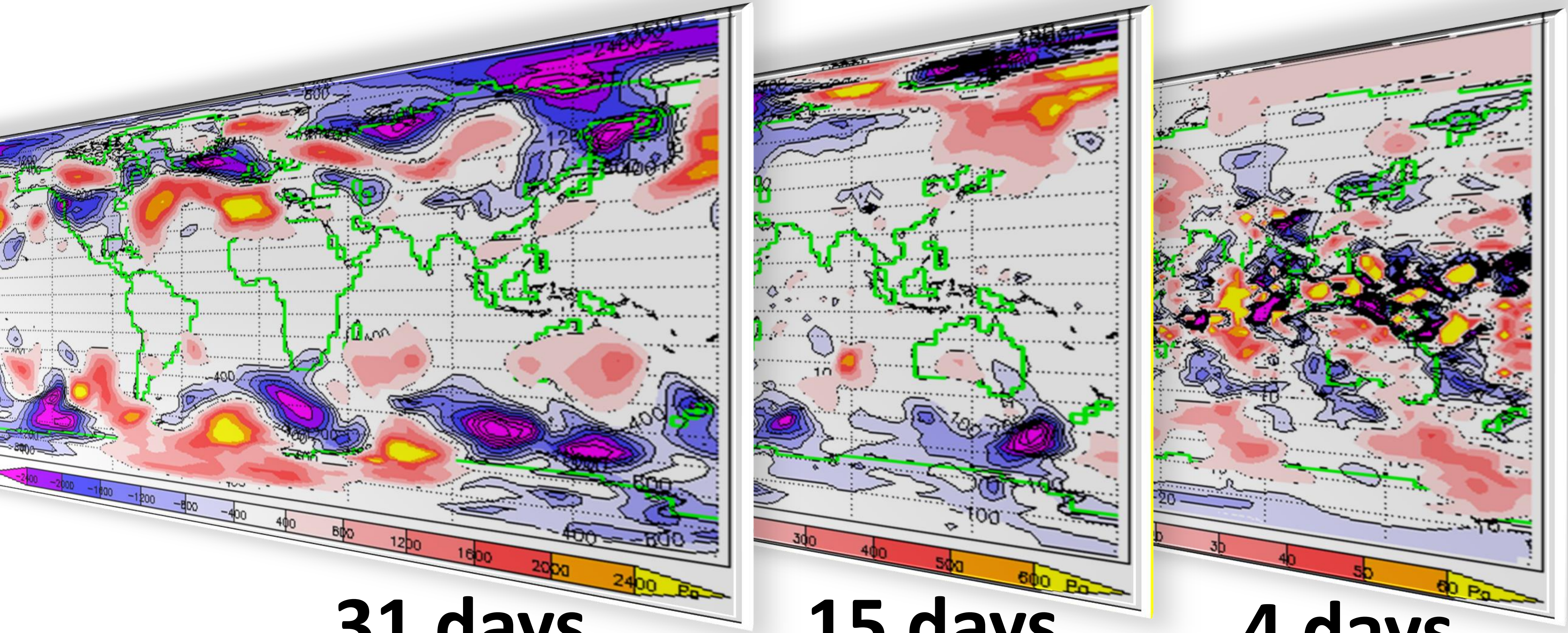
The climatic butterfly effect

Do numerical simulations capture the statistics of chaotic systems?

Qiqi Wang
Associate Professor
Aeronautics and Astronautics
Massachusetts Institute of Technology

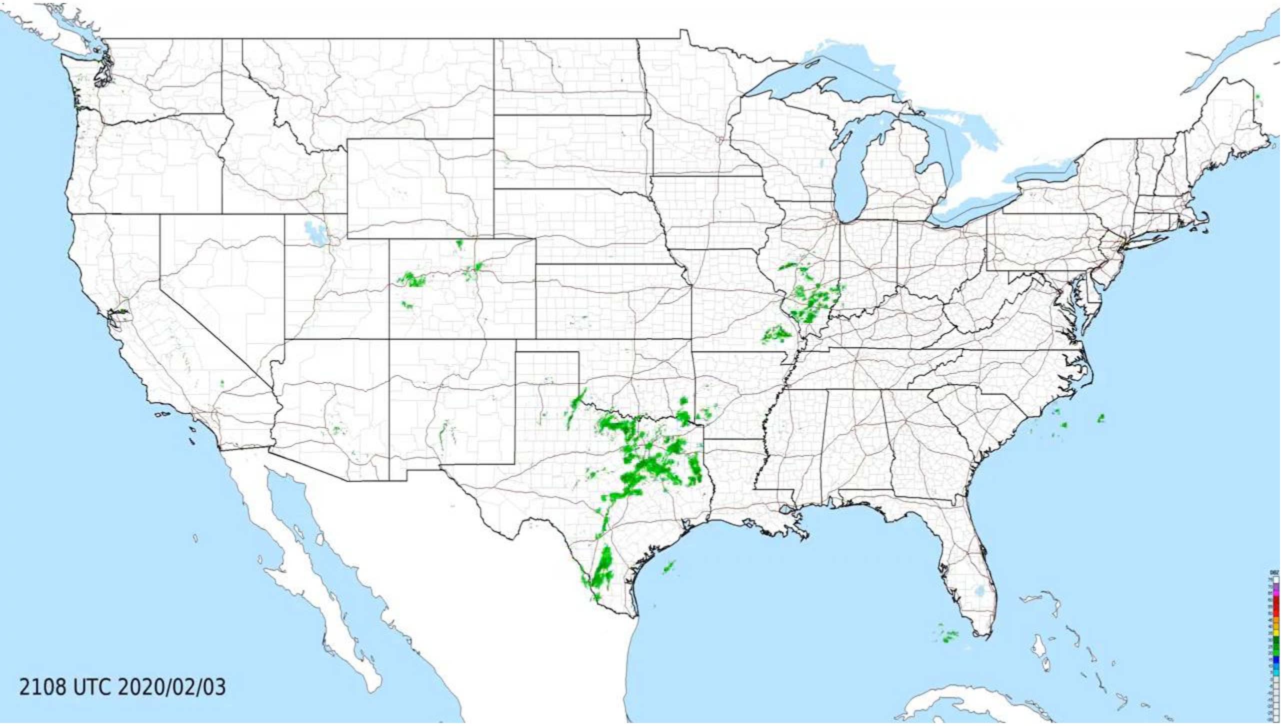
Butterfly effect



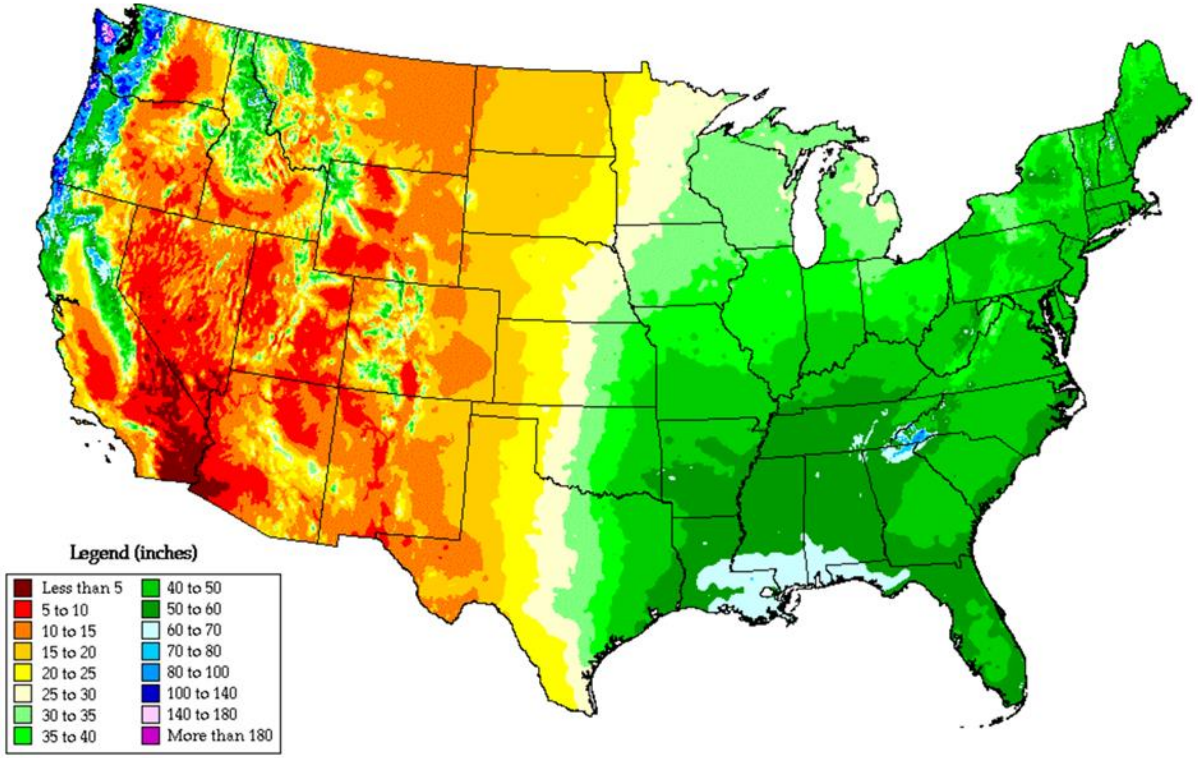
31 days **15 days** **4 days**

Sea Level Pressure Difference between perturbed simulations

Can a **butterfly** ~~cause a tornado~~?
control the climate?



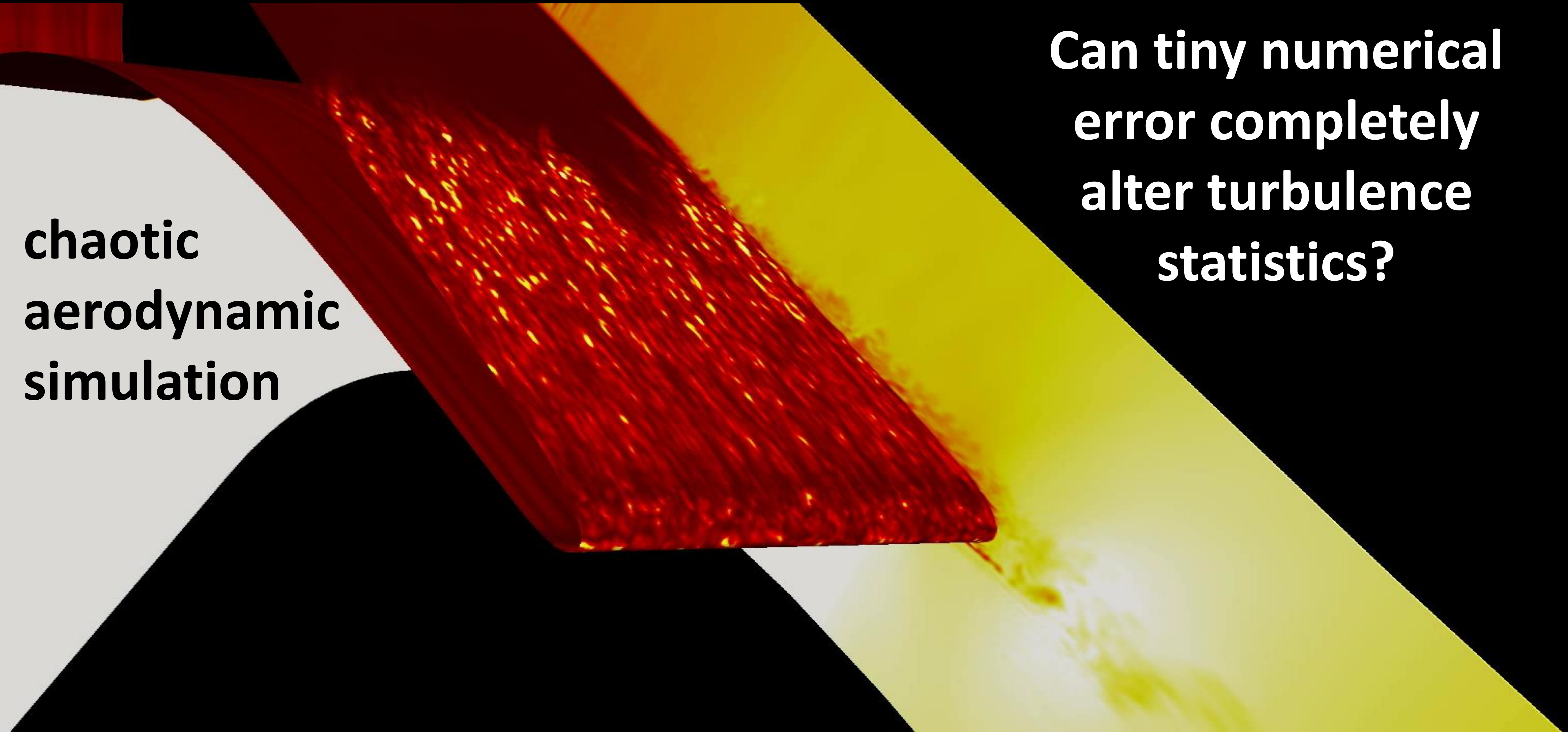
Feb 2020 10-minute precipitation



Annual precipitation 1961-1990

Climate: statistics of **weather** over a long time

What if a **butterfly** can **control the climate**?



Can a butterfly control the climate?

NO

- Ergodicity
- Shadowing

YES

- Ergodicity not applicable
- Shadowing nonphysical
- **Evidence in model system**

Can a butterfly control the climate?

NO

- Ergodicity
- Shadowing

YES

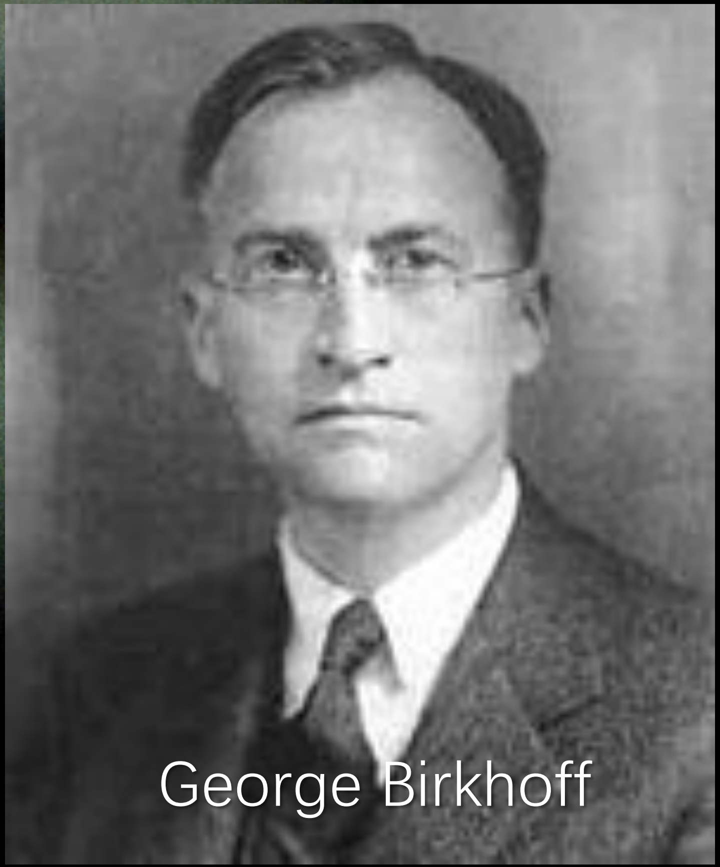
- Ergodicity not applicable
- Shadowing nonphysical
- **Evidence in model system**

Ergodicity

$$\langle J \rangle := \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T J(u(t))dt\right)$$

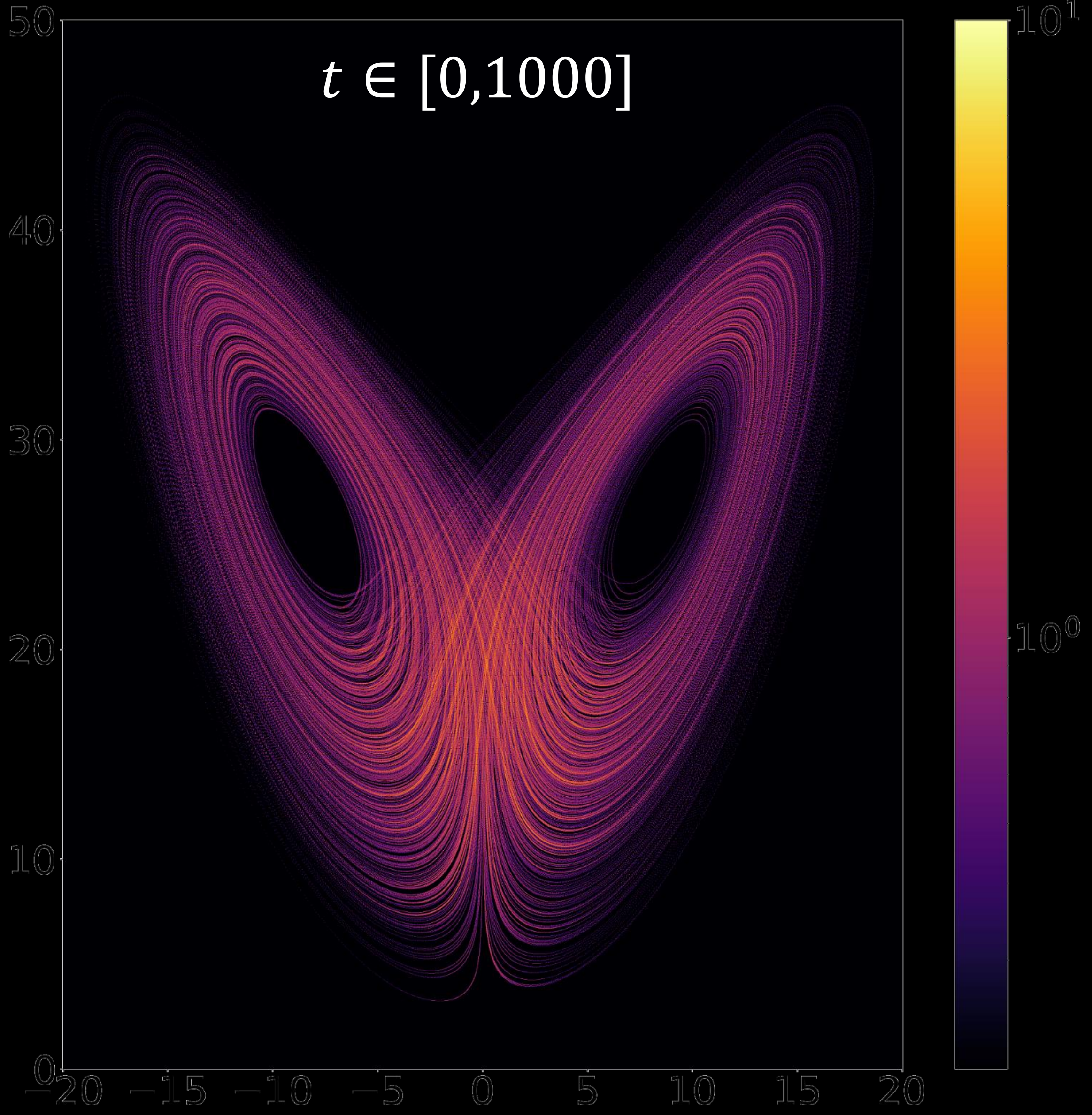
Henri Poincaré

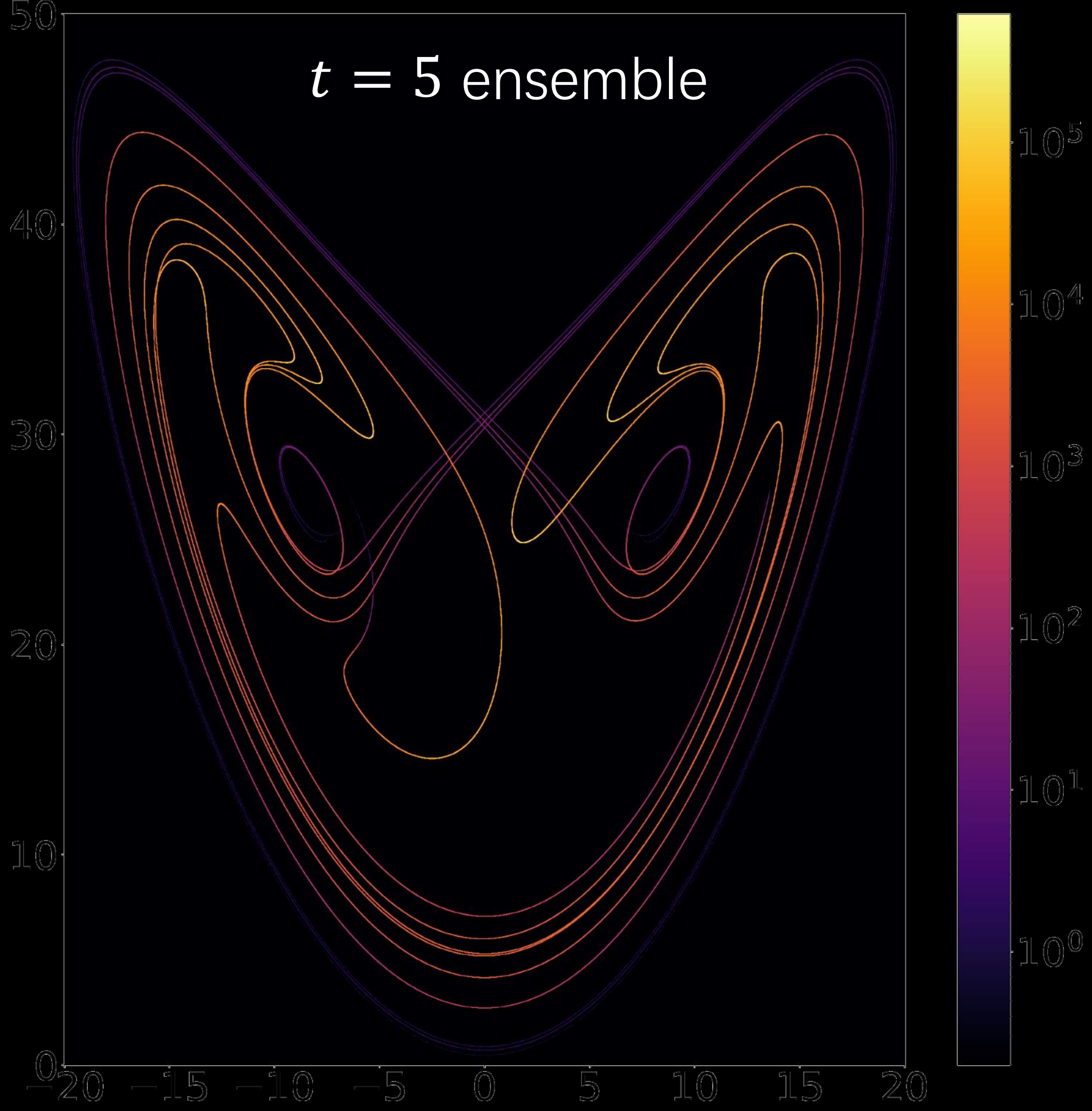
For a solution starting from almost any initial condition,
the **time average** of a function of the solution equals the **ensemble average**



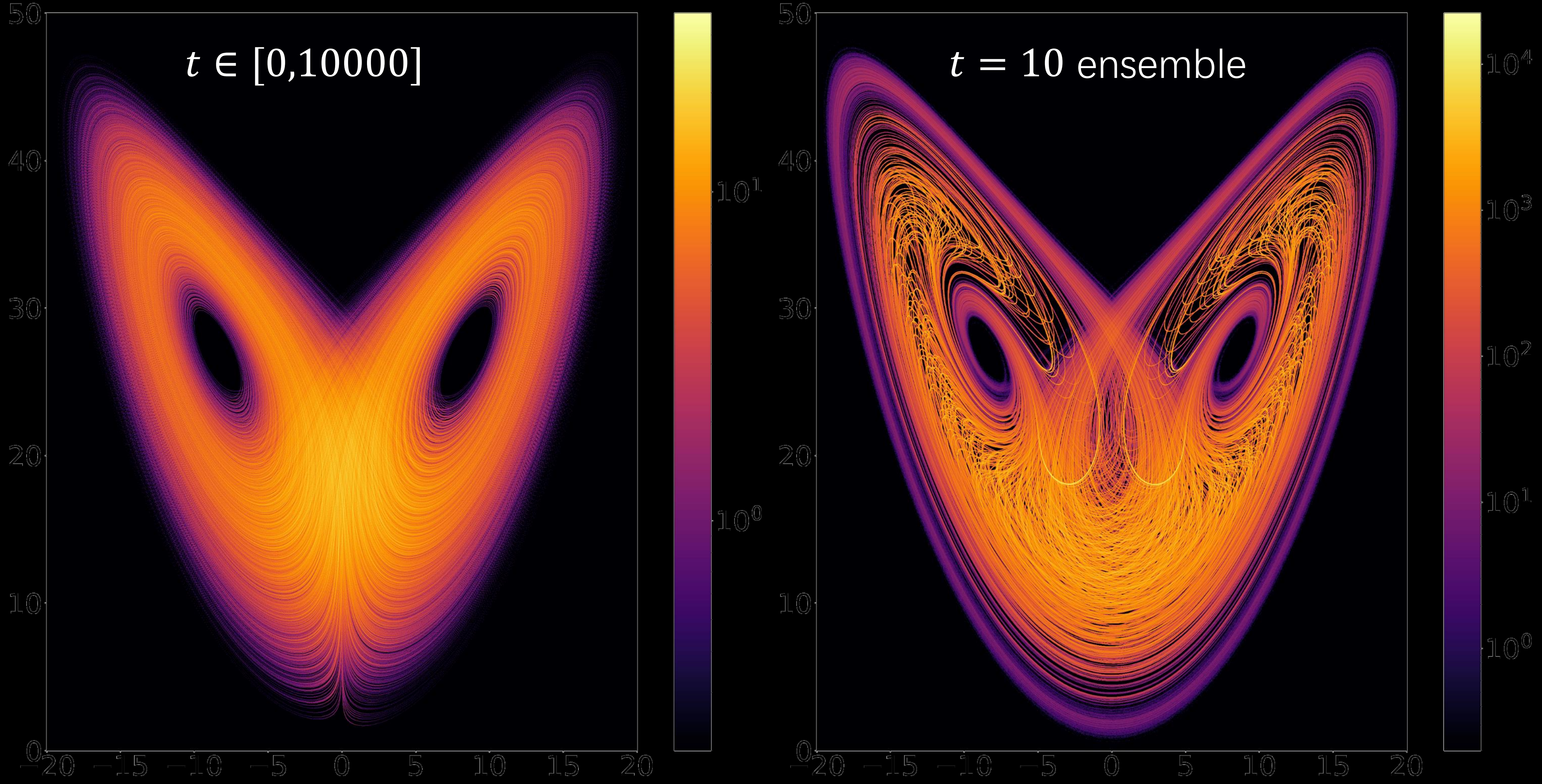
George Birkhoff

Ergodicity


$t \in [0,1000]$
50
40
30
20
10
0
−20
−15
−10
−5
0
5
10
15
20
10^1
10^0

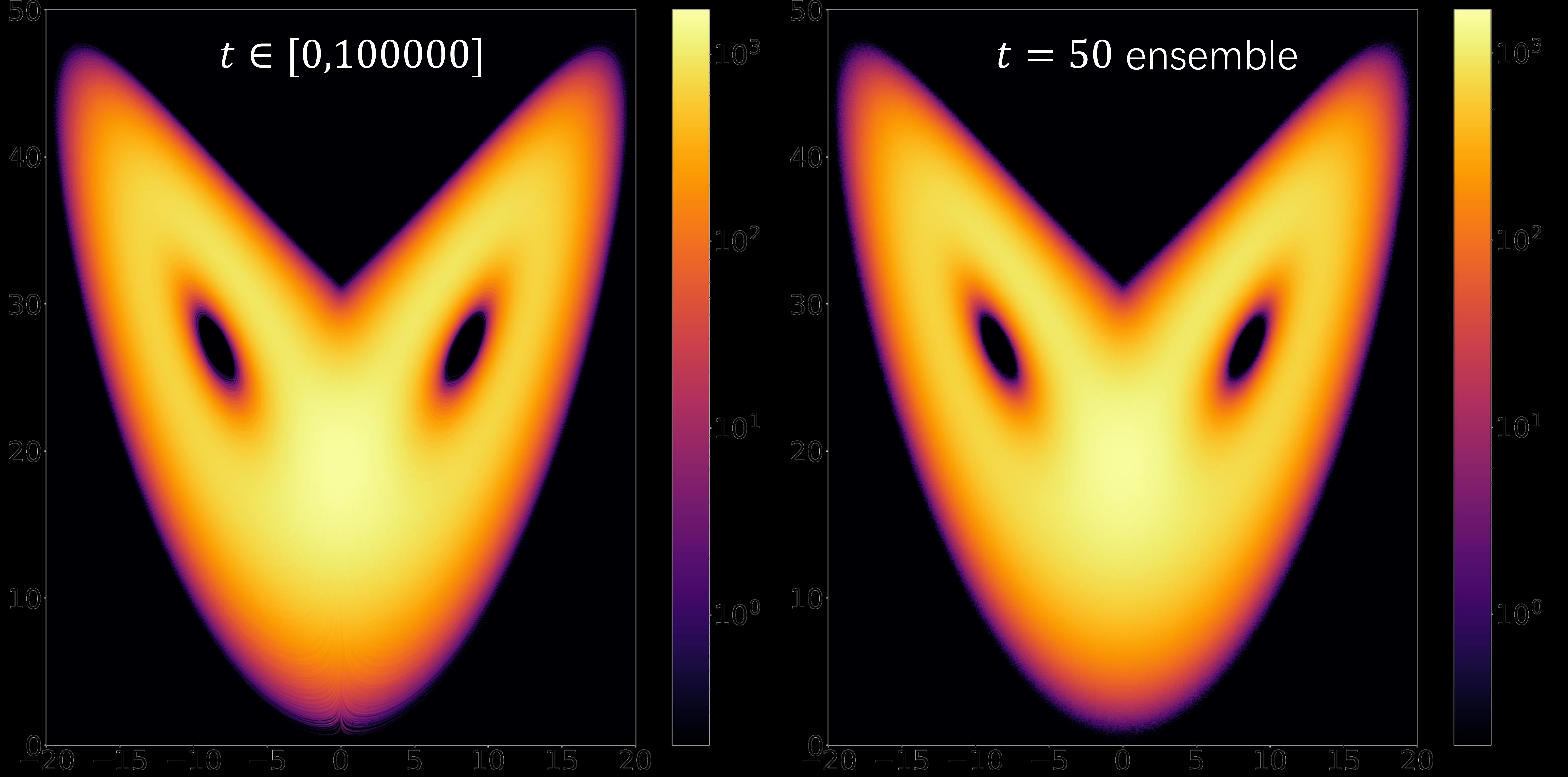

$t = 5$ ensemble
50
40
30
20
10
0
−20
−15
−10
−5
0
5
10
15
20
10^5
10^4
10^3
10^2
10^1
10^0

Ergodicity



$t \in [0,10000]$
$t = 10$ ensemble
50
40
30
20
10
0
−20
−15
−10
−5
0
5
10
15
20
10^1
10^0
10^4
10^3
10^2
10^1
10^0

Ergodicity



$t \in [0,100000]$
$t = 50$ ensemble
0
10
20
30
40
50
−20
−15
−10
−5
0
5
10
15
20
10^0
10^1
10^2
10^3

Can a butterfly control the climate?

NO

- Ergodicity
- Shadowing

YES

- **Ergodicity not applicable**
- Shadowing nonphysical
- **Evidence in model system**

Ergodicity not applicable

Different initial condition, same statistics
Climate unchanged by a **one-time** perturbation

Can tiny but **persistent** perturbation significantly modify the statistics?

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}) + \boldsymbol{\delta f}(\boldsymbol{u})$$

Can a butterfly control the climate?

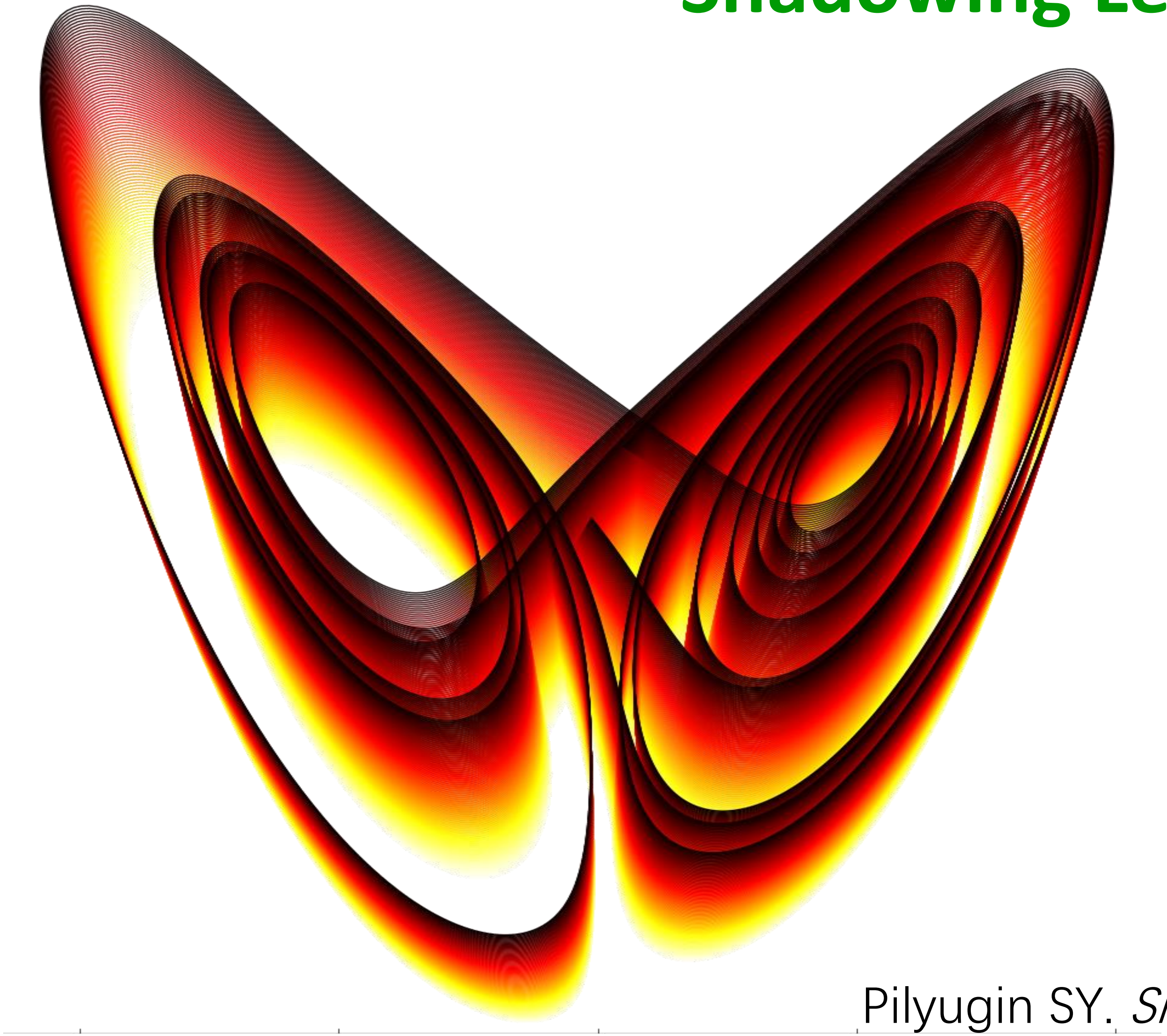
NO

- Ergodicity
- **Shadowing**

YES

- Ergodicity not applicable
- Shadowing nonphysical
- **Evidence in model system**

Shadowing Lemma



$\forall \epsilon > \mathbf{0}, \exists \delta$, s.t.

$\forall v(\mathrm{t})$ s.t.

$$\frac{dv}{dt} = f(v) + \delta f$$

$$\|\delta f\| < \delta$$

$\exists u(t)$ s.t.

$$\frac{du}{dt} = f(u)$$

$$\|u - v\| < \epsilon$$

Pilyugin SY. *Shadowing in dynamical systems* 1999

Starting from same initial condition

Shadowing solutions

Can a **butterfly** **control the climate**?

No, because of shadowing:

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{\delta f}(\boldsymbol{v})$$

Small perturbation $\boldsymbol{\delta f}(\boldsymbol{v})$ means a shadowing solution exists:

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u})$$

with $\|\boldsymbol{u} - \boldsymbol{v}\| < \epsilon$. The statistics of $\boldsymbol{v}$ (perturbed solution) and $\boldsymbol{u}$ (unperturbed) are therefore close.

Can a **butterfly control the climate**?

NO

- Ergodicity
- Shadowing

YES

- Ergodicity not applicable
- **Shadowing nonphysical**
- **Evidence in model system**

Can a butterfly control the climate?

No, because of shadowing:

$$\frac{dv}{dt} = f(v) + \delta f(v)$$

Small perturbation $\delta f(v)$ means a shadowing solution exists:

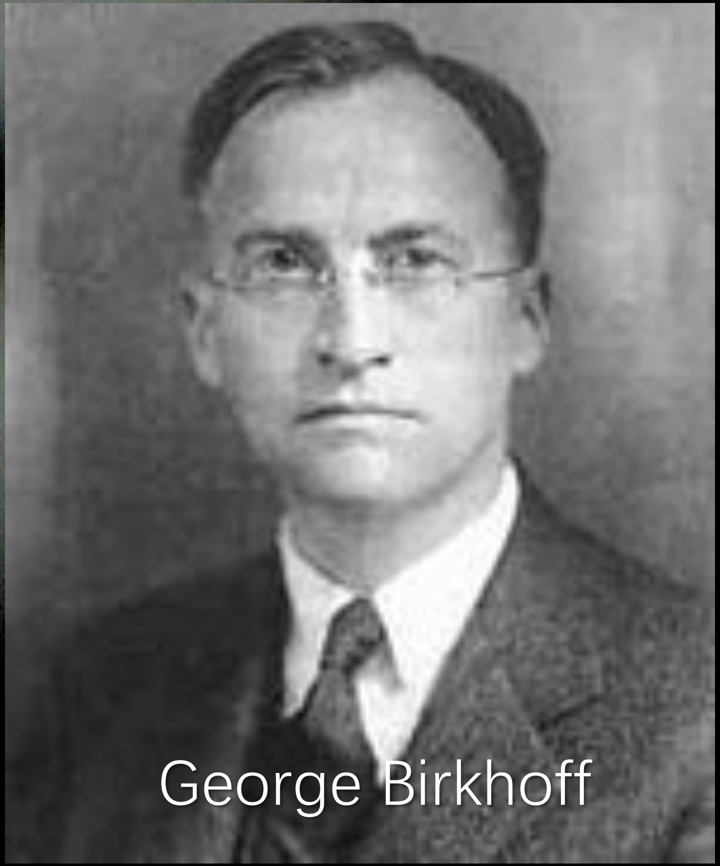
$$\frac{du}{dt} = f(u)$$

with $\|u - v\| < \epsilon$. The statistics of v (perturbed solution) and u (unperturbed) are therefore close.

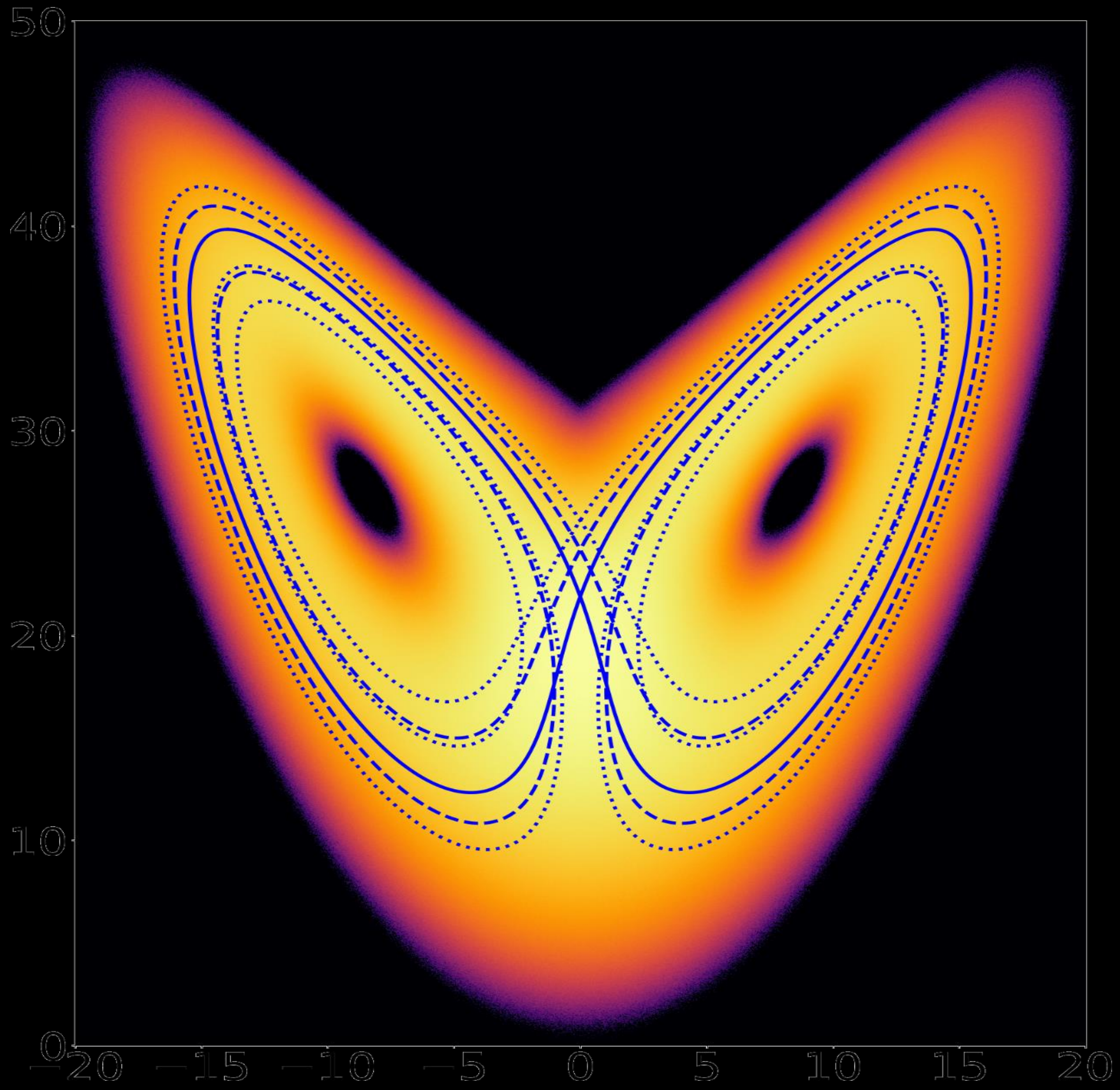
Ergodicity

$$\langle J \rangle := \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T J(u(t))dt\right)$$

For a solution starting from **almost** any initial condition, the **time average** of a function of the solution equals the **ensemble average**



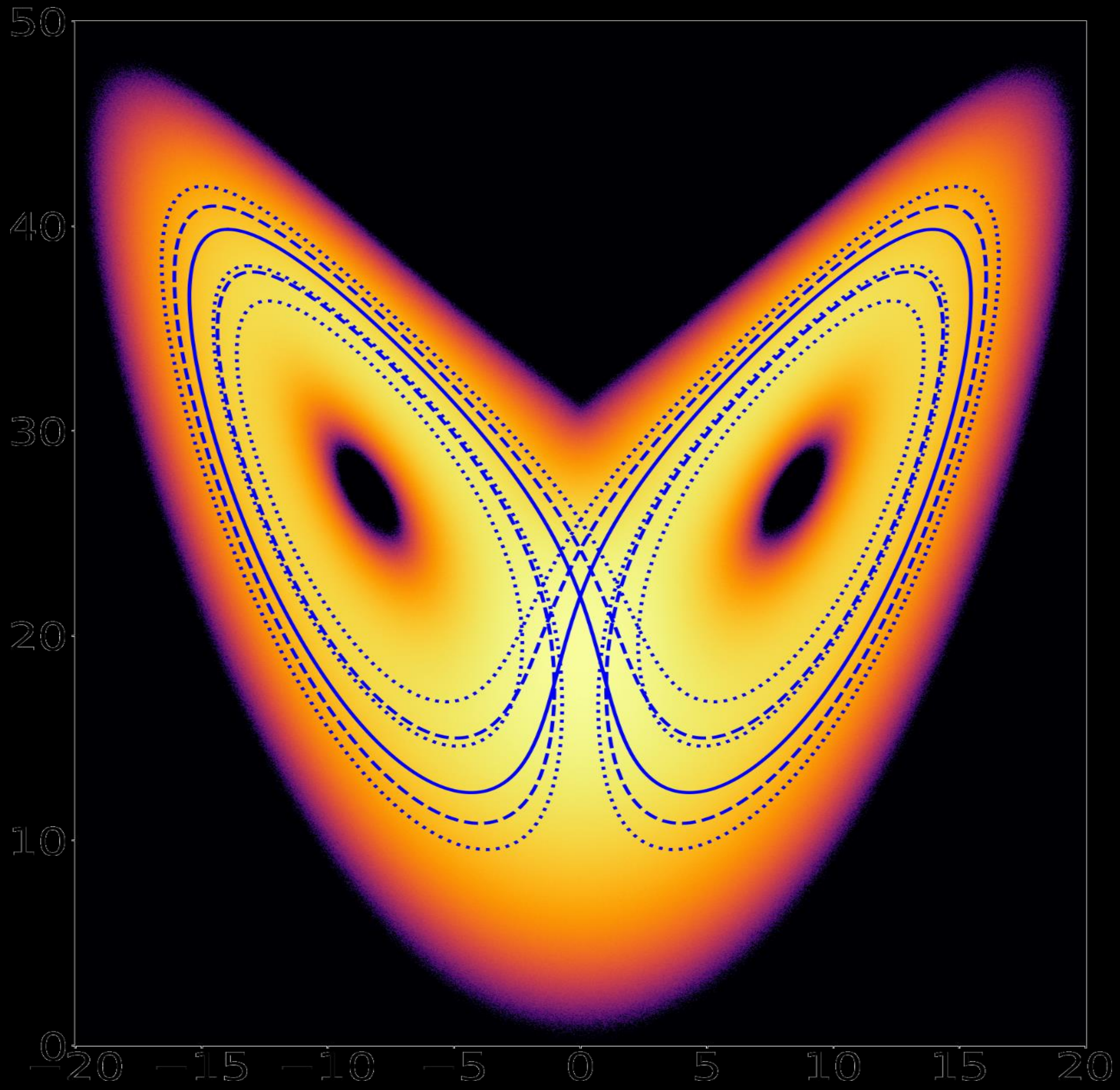
Nonphysical solutions

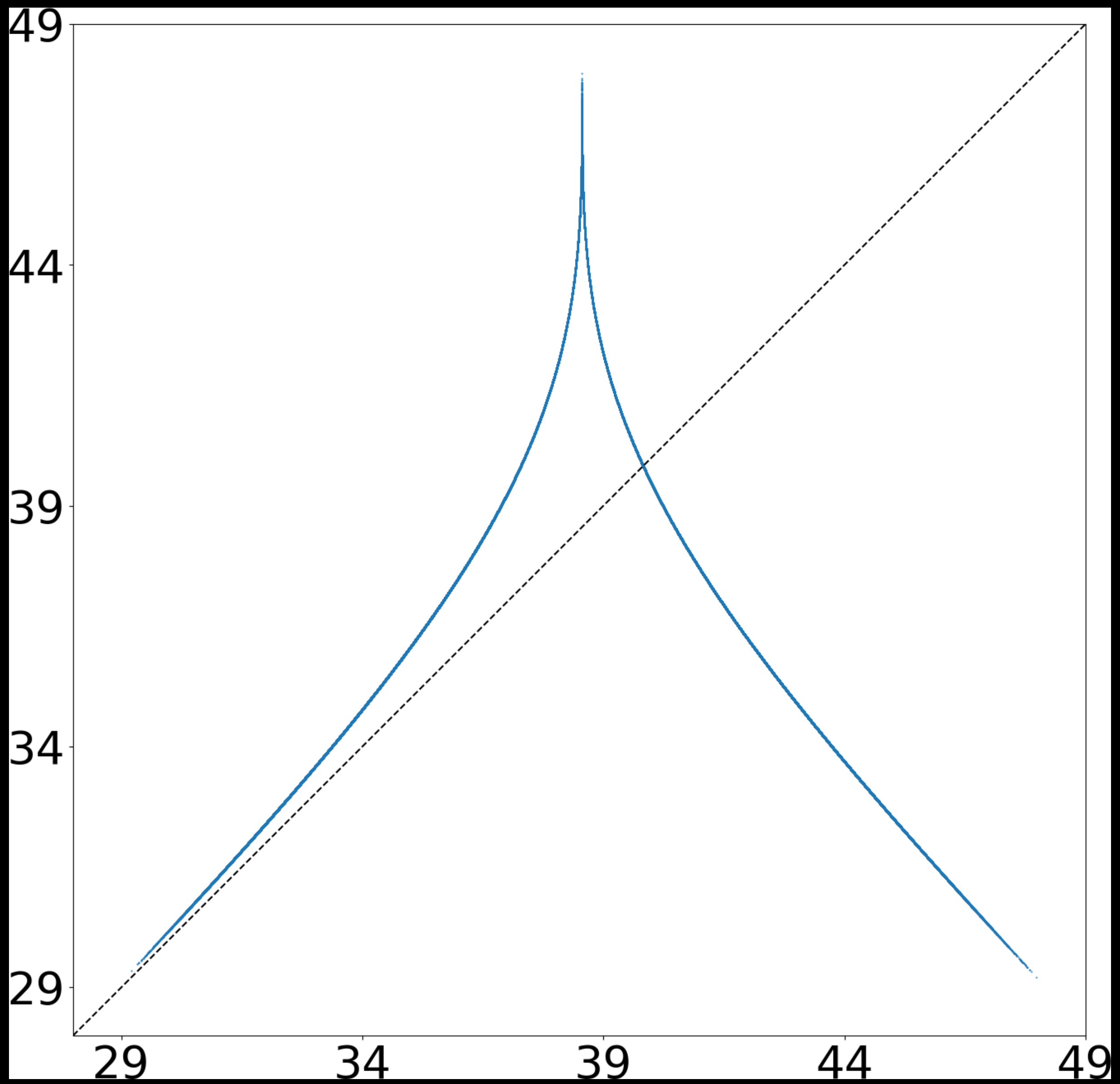


Starting from an unlikely (measure-zero) set of initial conditions, time average differs from ensemble average

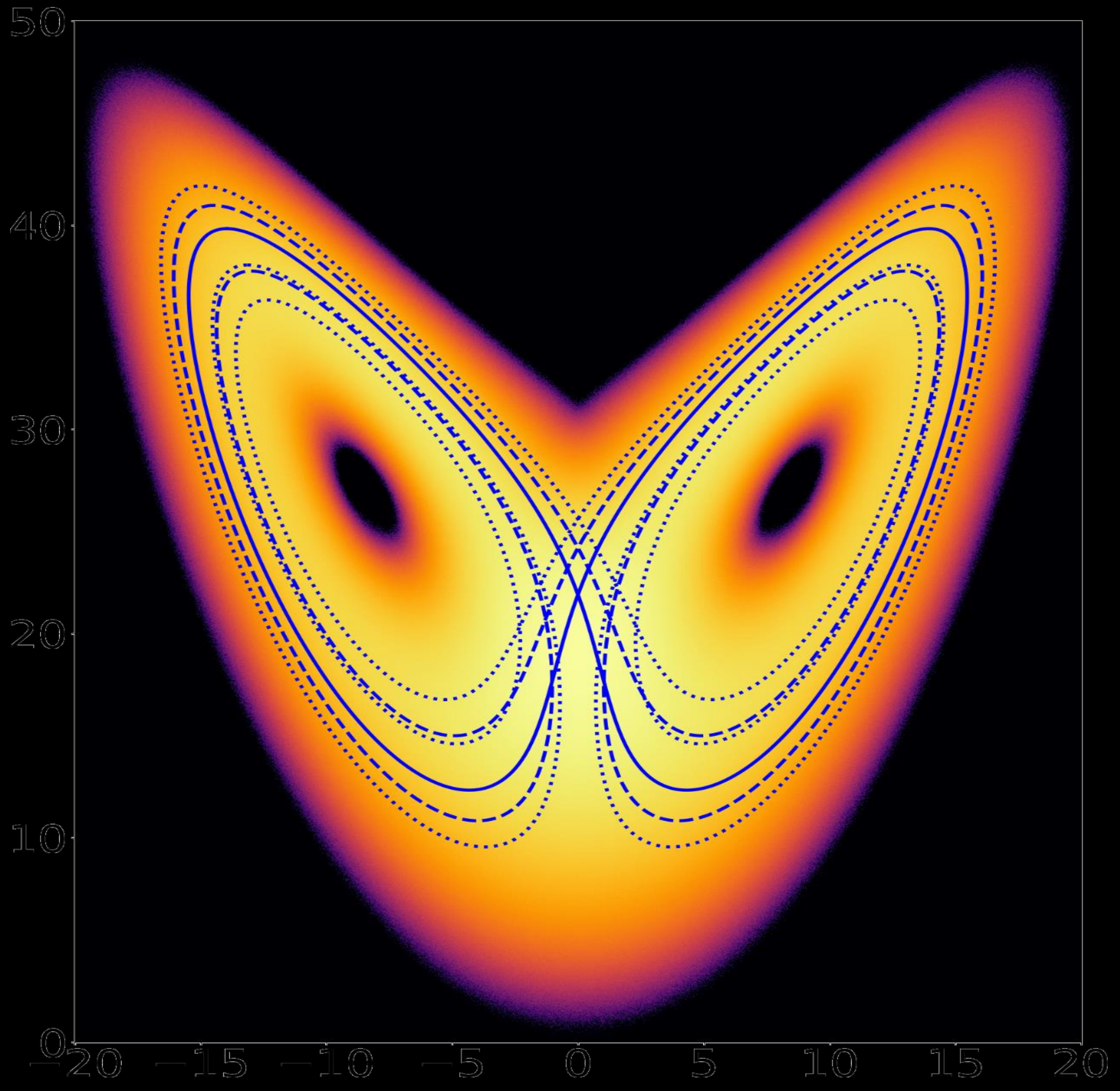
1. **Periodic solutions**
2. Quasi-physical solutions

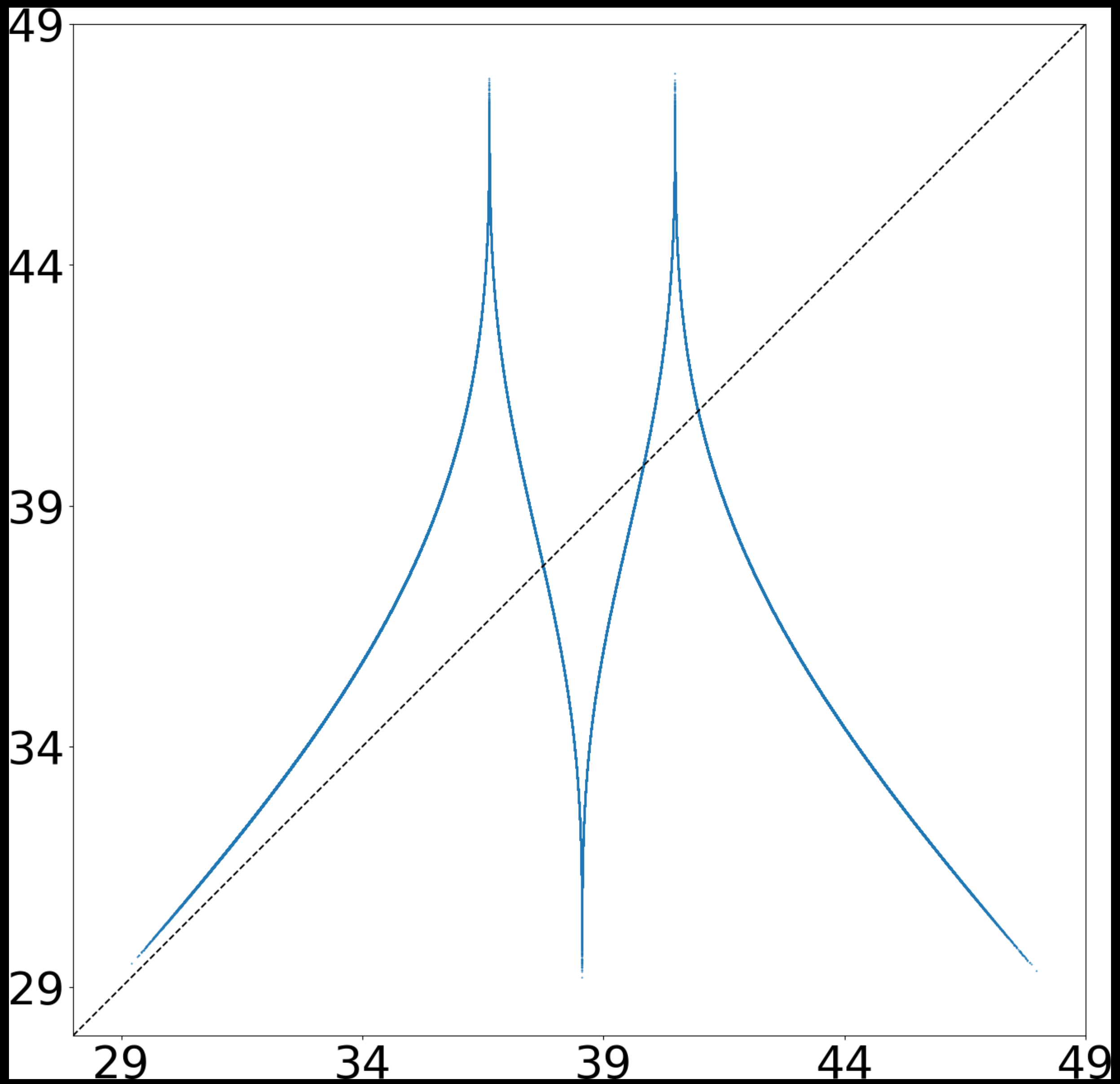
1. Periodic nonphysical solutions



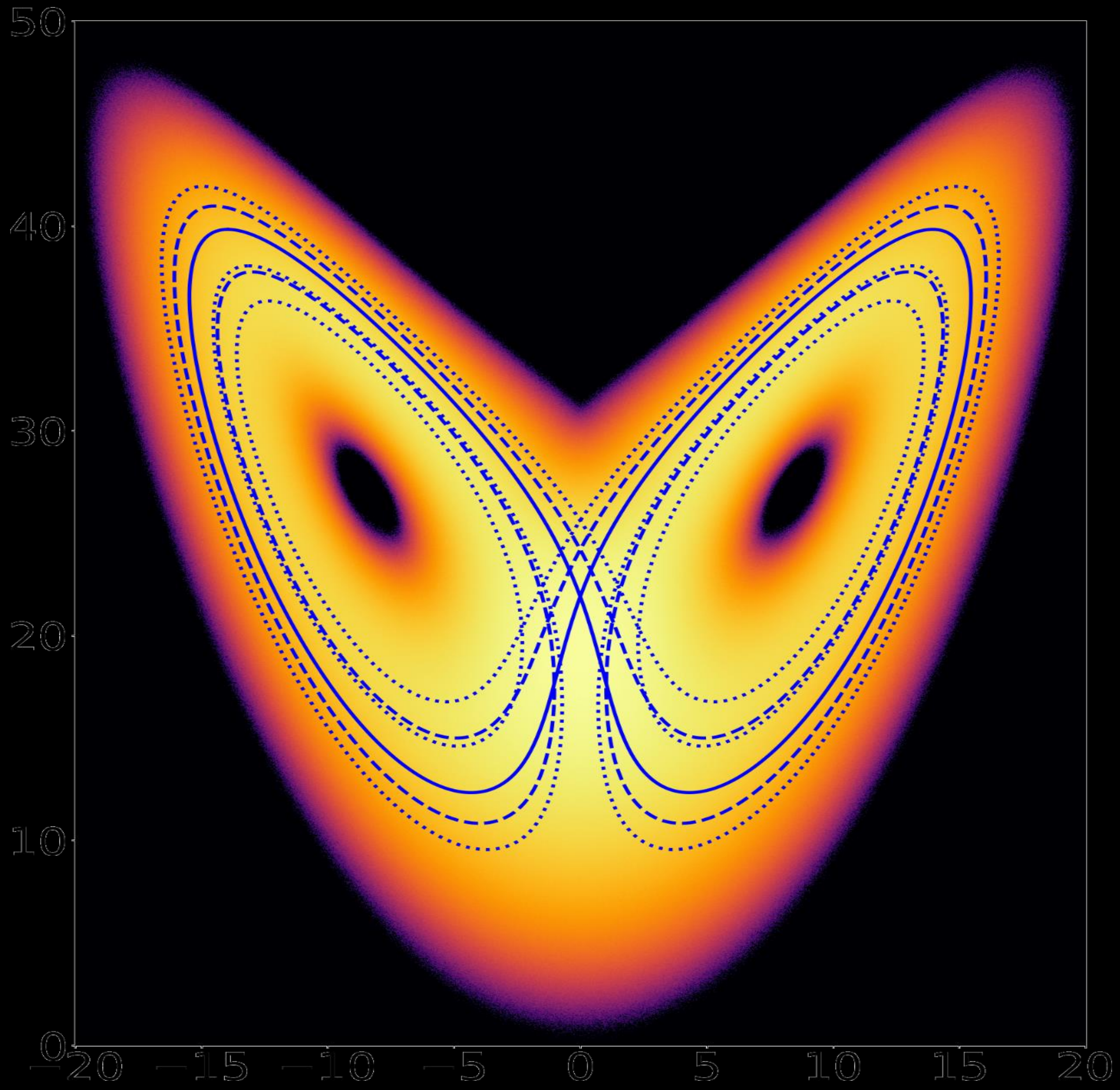


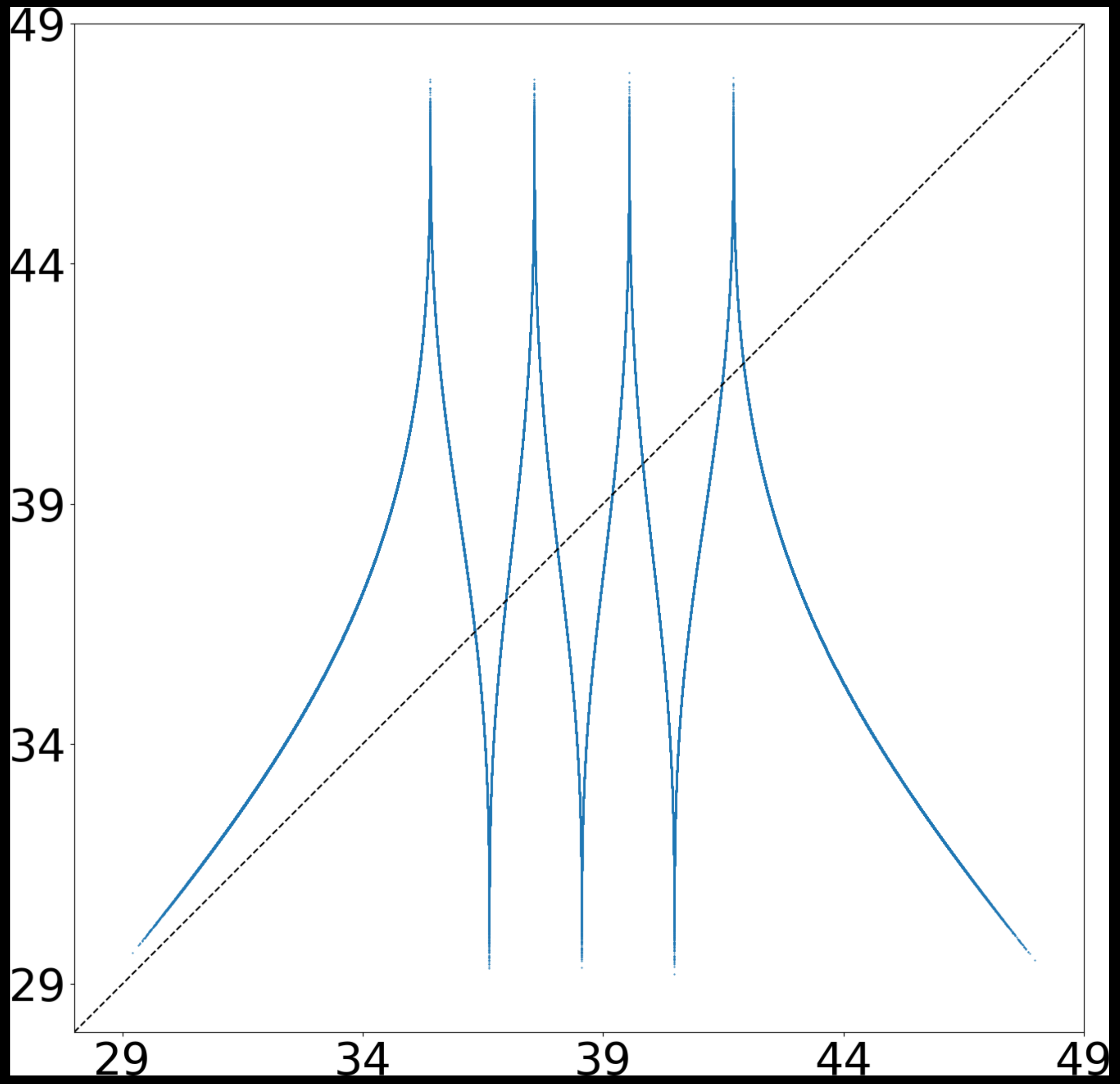
1. Periodic nonphysical solutions



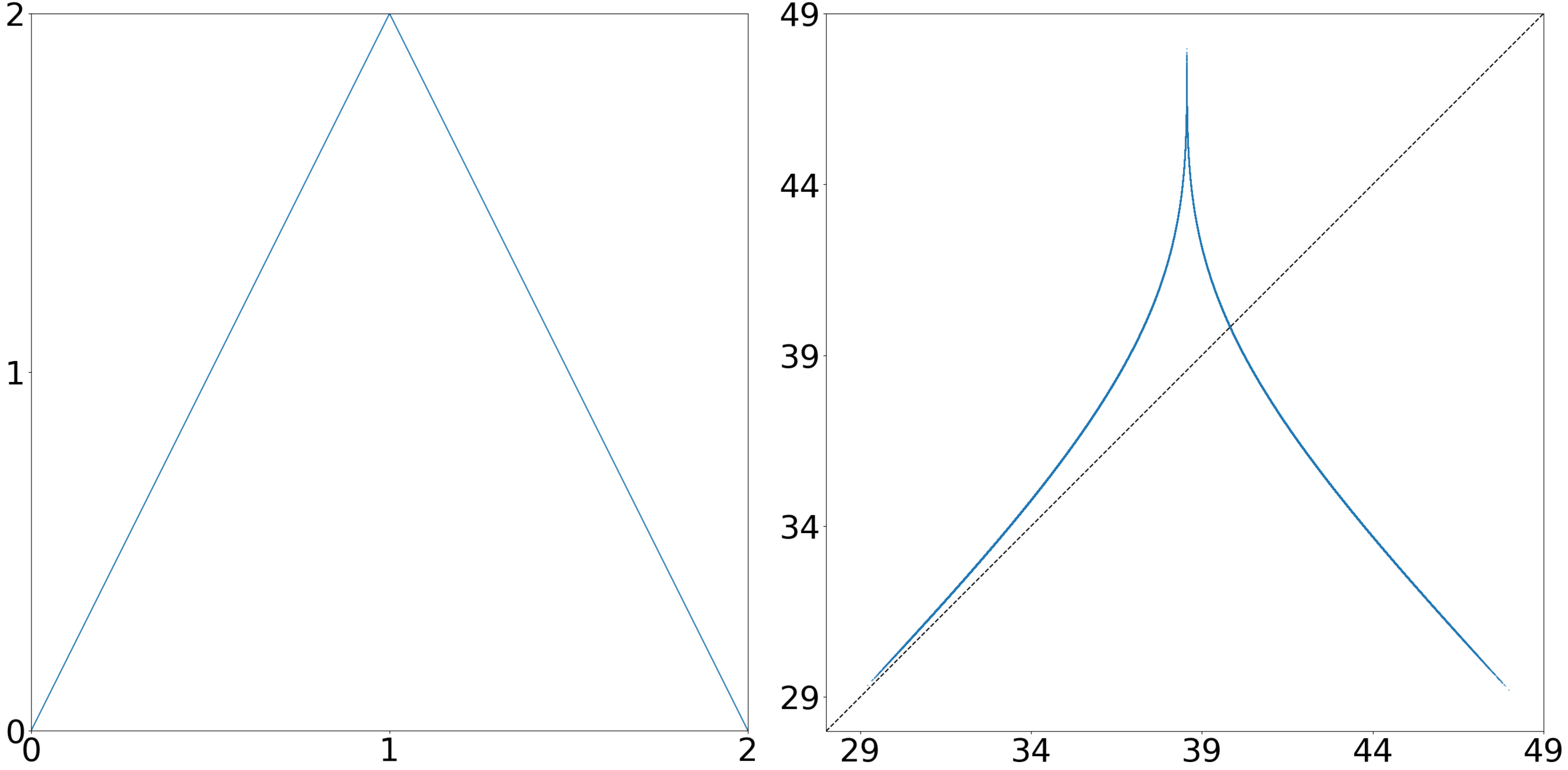


1. Periodic nonphysical solutions

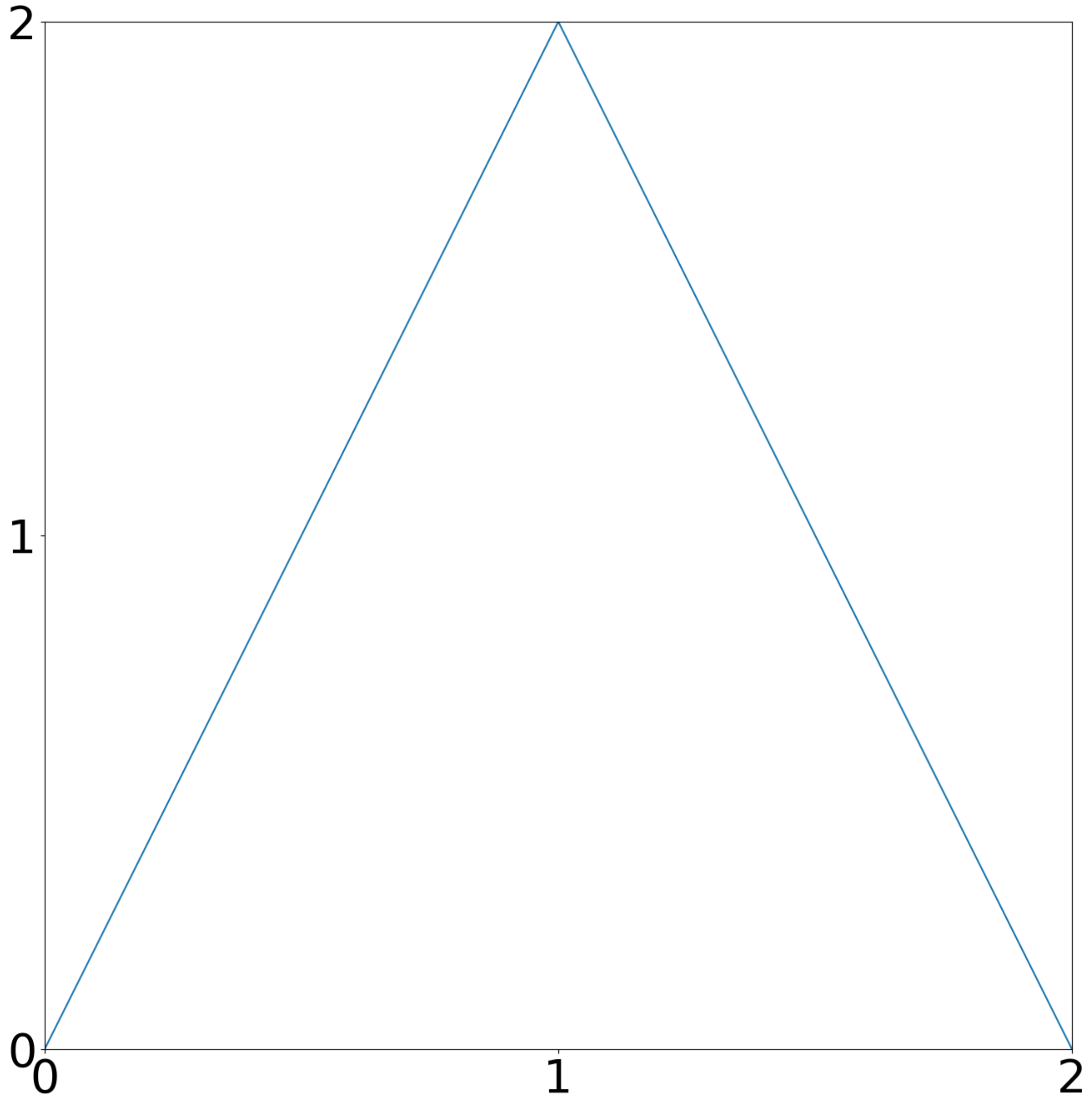


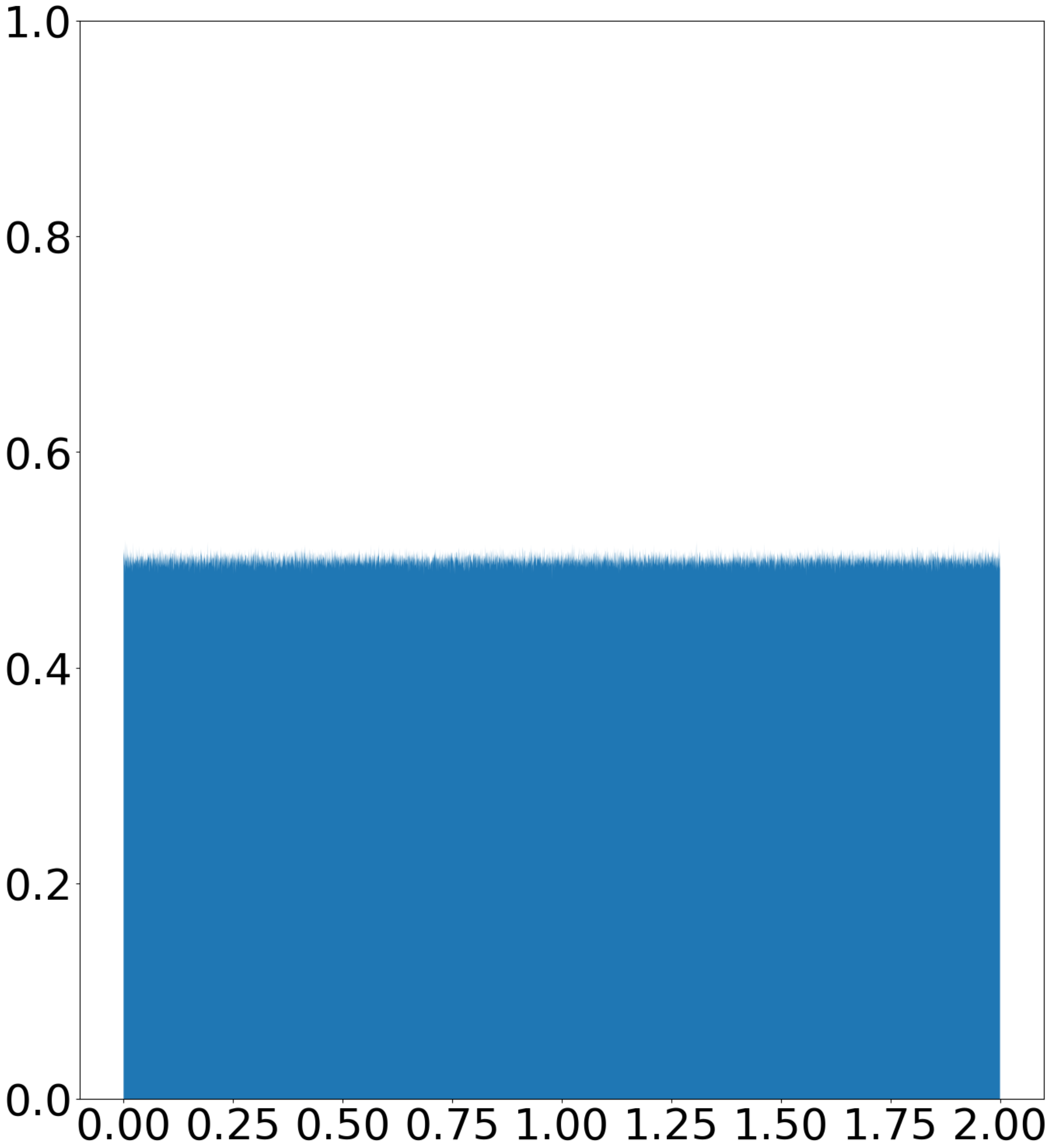


Simplified the Lorenz map: tent map

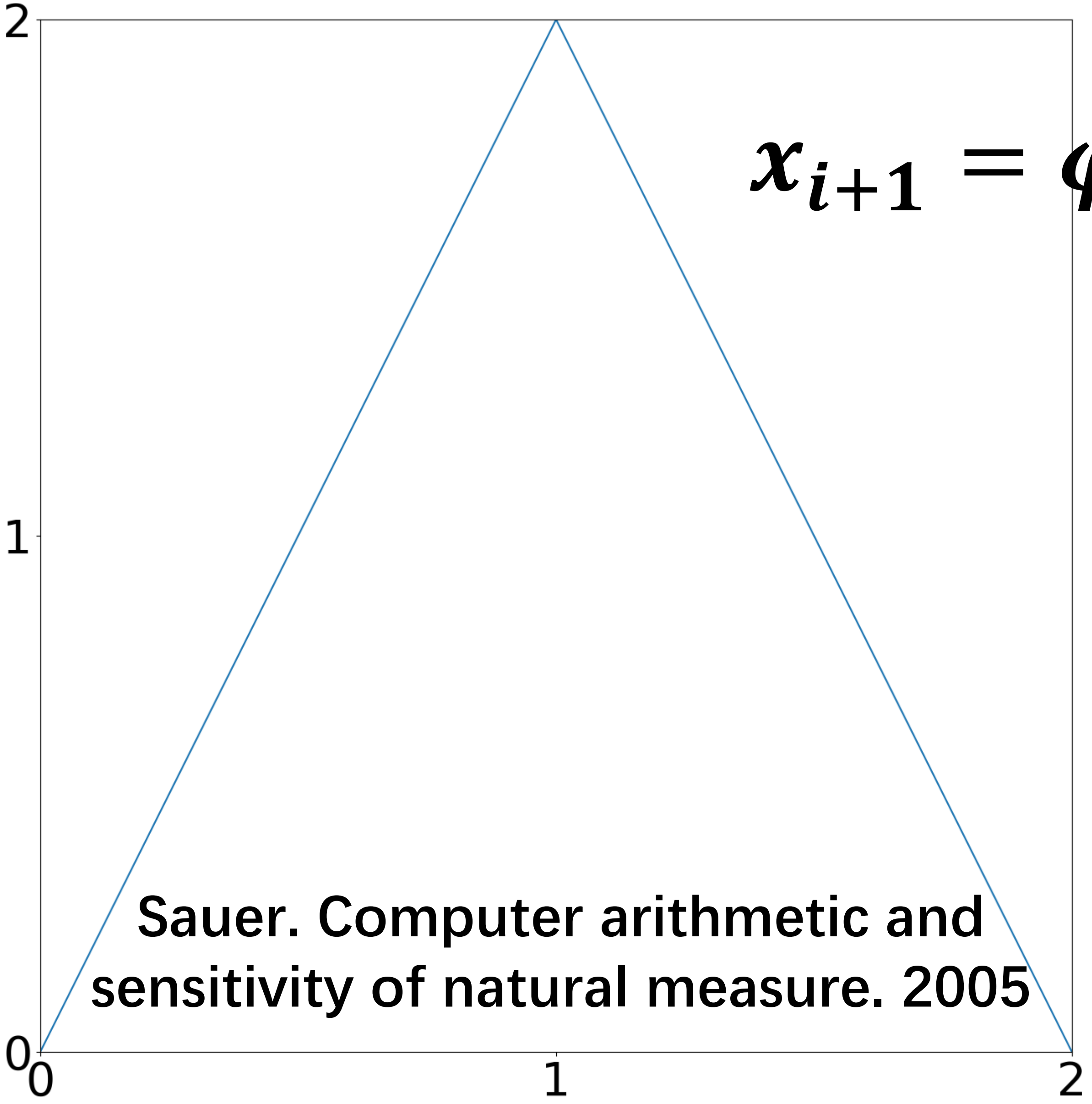


Tent map: physical solutions





Tent map: Periodic nonphysical solutions



Sauer. Computer arithmetic and sensitivity of natural measure. 2005

$$x_{i+1} = \varphi(x_i) = \begin{cases} 2x_i, & x_i < 1 \\ 2(2 - x_i), & x_i \geq 1 \end{cases}$$

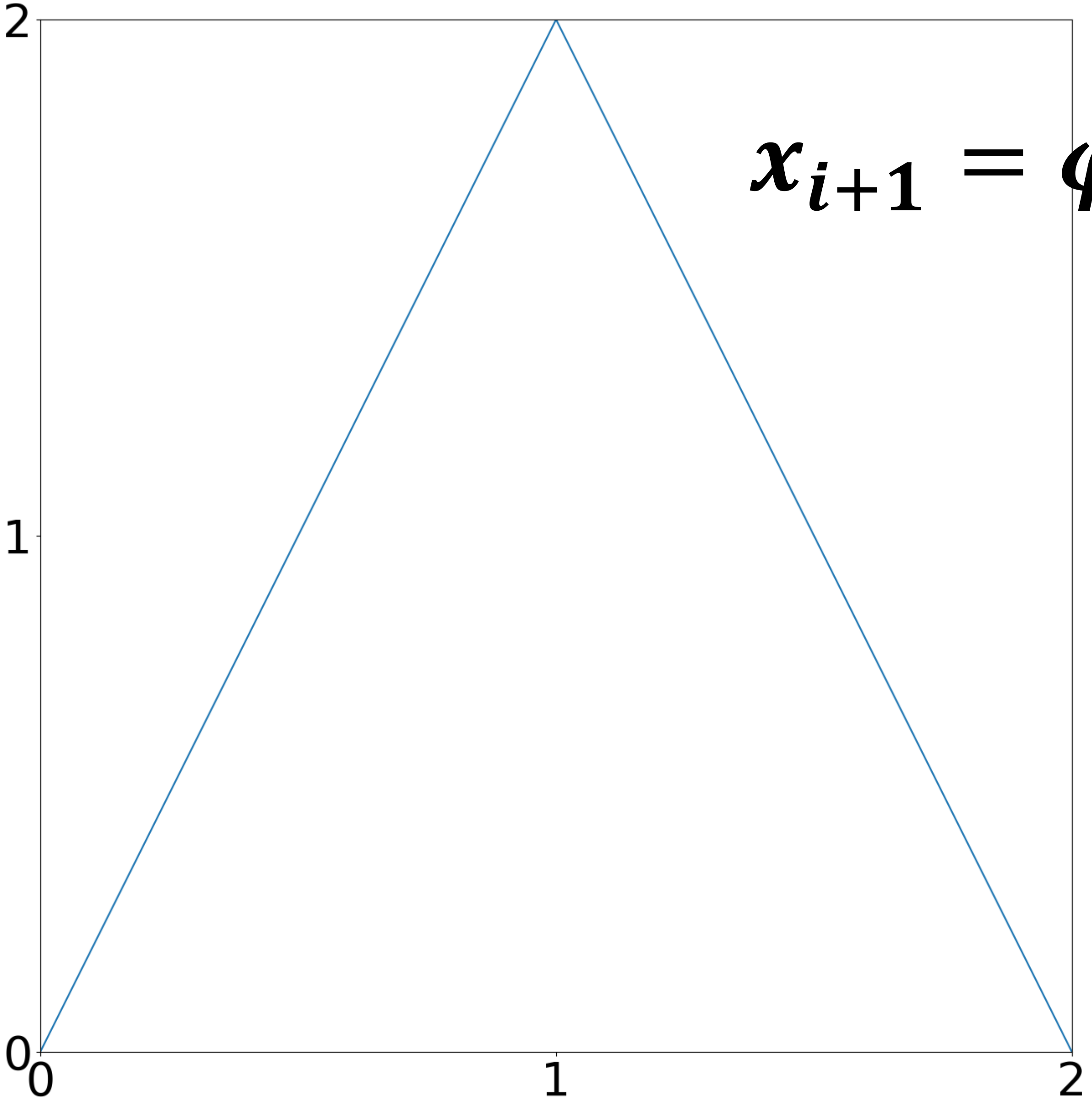
Represent

$$x_i = \sum_{k=0}^{\infty} \frac{x_i^{(k)}}{2^k}$$

Then,

$$x_{i+1}^{(k)} = x_i^0 \ xor \ x_i^{k+1}$$

Tent map: Periodic nonphysical shadowing solutions



$$x_{i+1} = \varphi(x_i) = \begin{cases} 2x_i, & x_i < 1 \\ 2(2 - x_i), & x_i \geq 1 \end{cases}$$

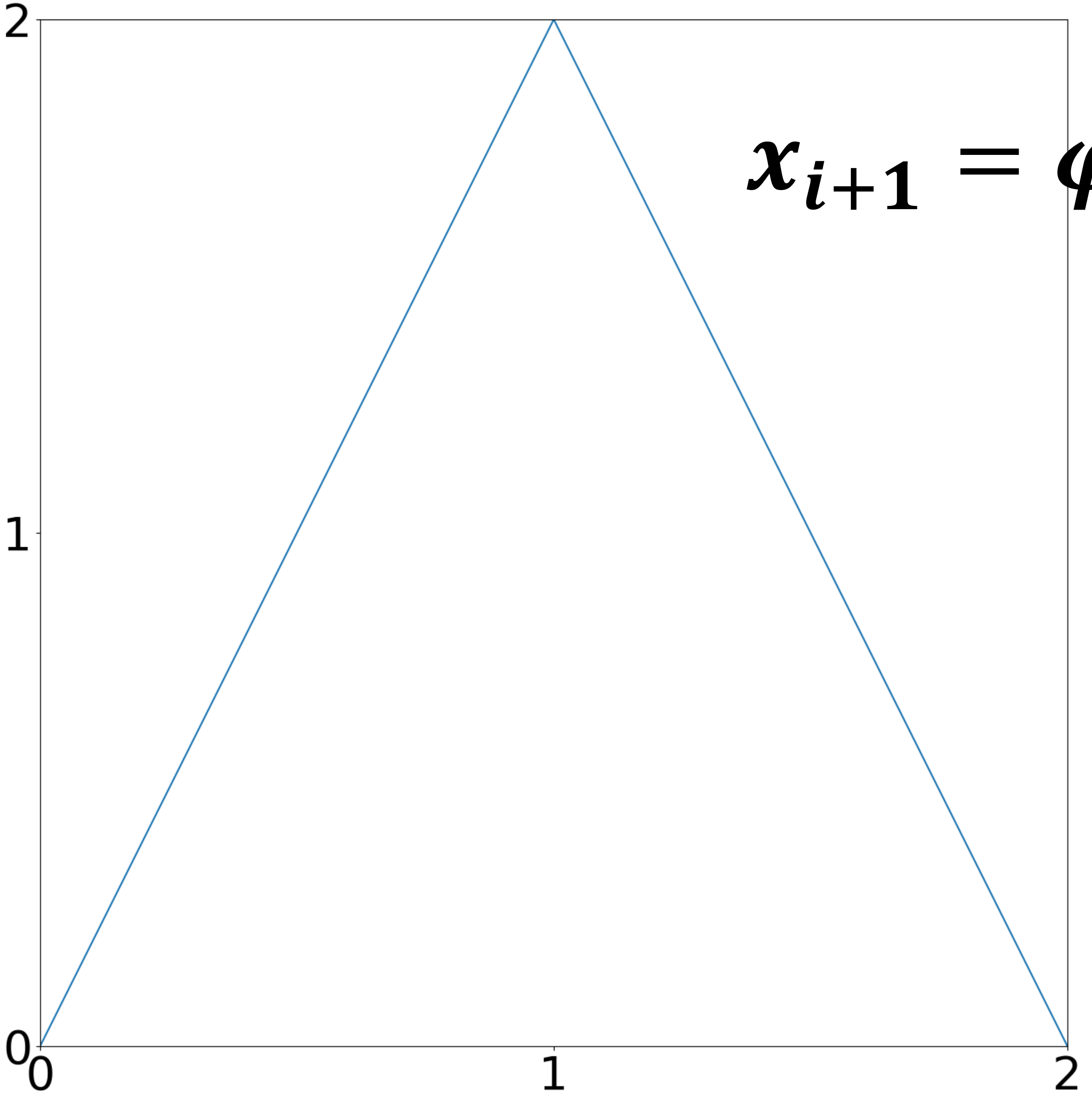
Represent

$$x_i = \sum_{k=0}^{\infty} \frac{x_i^{(k)}}{2^k}$$

Then,

$$x_{i+1}^{(k)} = x_i^0 \; xor \; x_i^{k+1}$$

Tent map: Quasi-physical solutions



$$x_{i+1} = \varphi(x_i) = \begin{cases} 2x_i, & x_i < 1 \\ 2(2 - x_i), & x_i \geq 1 \end{cases}$$

Represent

$$x_i = \sum_{k=0}^{\infty} \frac{x_i^{(k)}}{2^k}$$

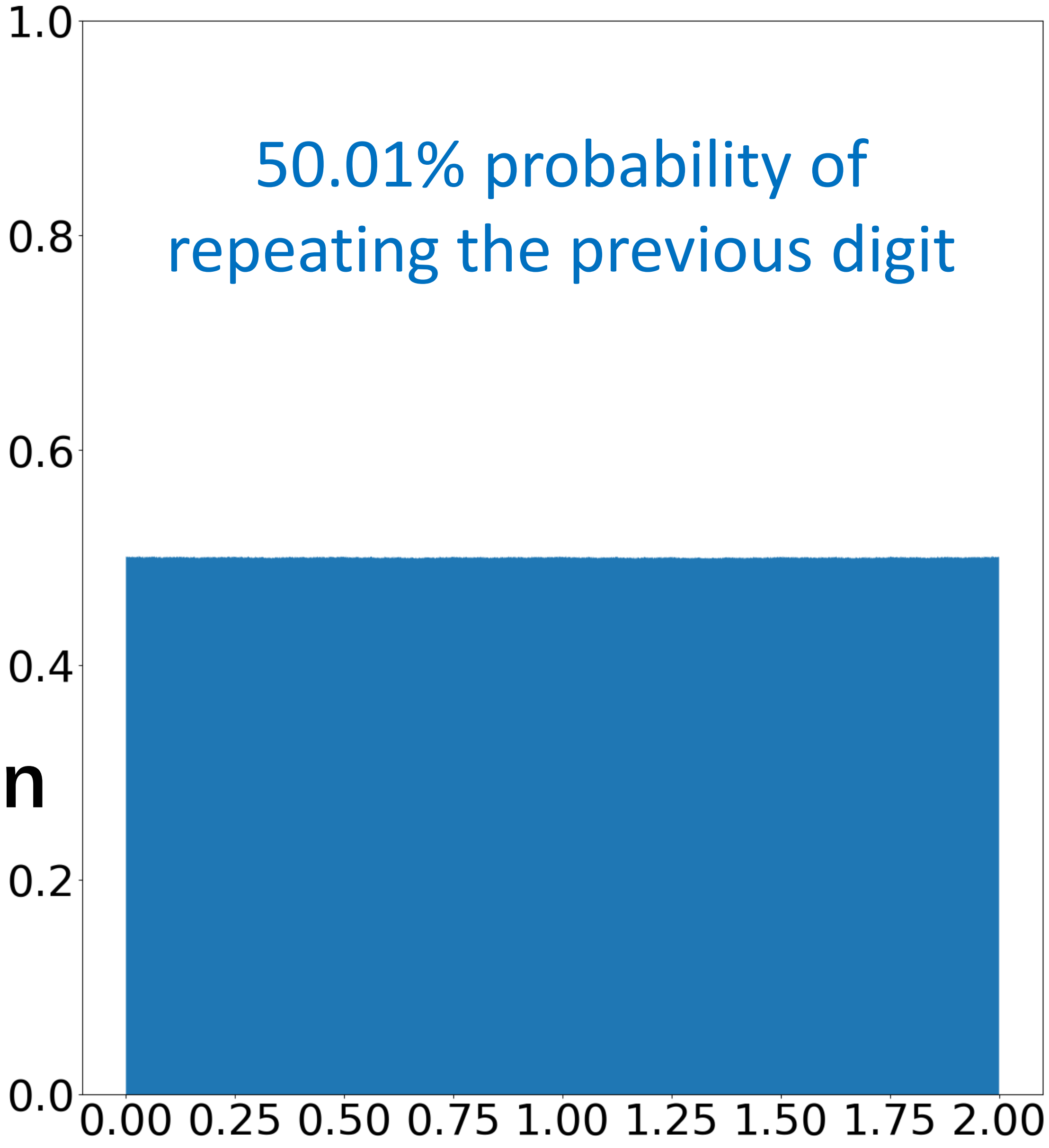
Then,

$$x_{i+1}^{(k)} = x_i^0 \;\; xor \;\; x_i^{k+1}$$

Tent map: Quasi-physical solutions

$$x_i = \sum_{k=0}^{\infty} \frac{x_i^{(k)}}{2^k}$$

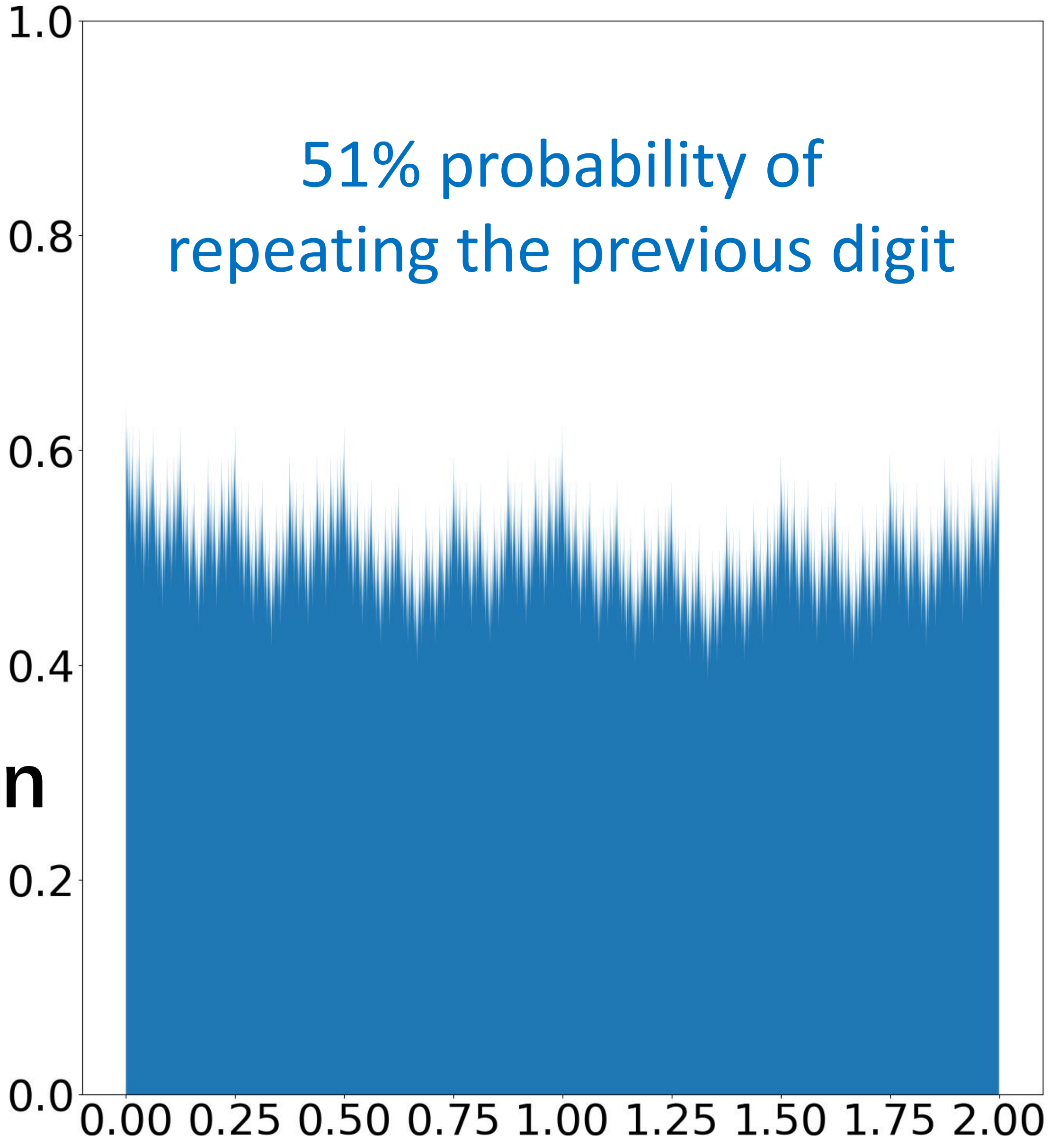
- Physical solution: $x_i^{(k)}$ i.i.d
- Periodic: $x_i^{(k)}$ determined by previous digits
- Quasi-physical: $x_i^{(k)}$ depends on previous digits



Tent map: Quasi-physical solutions

$$x_i = \sum_{k=0}^{\infty} \frac{x_i^{(k)}}{2^k}$$

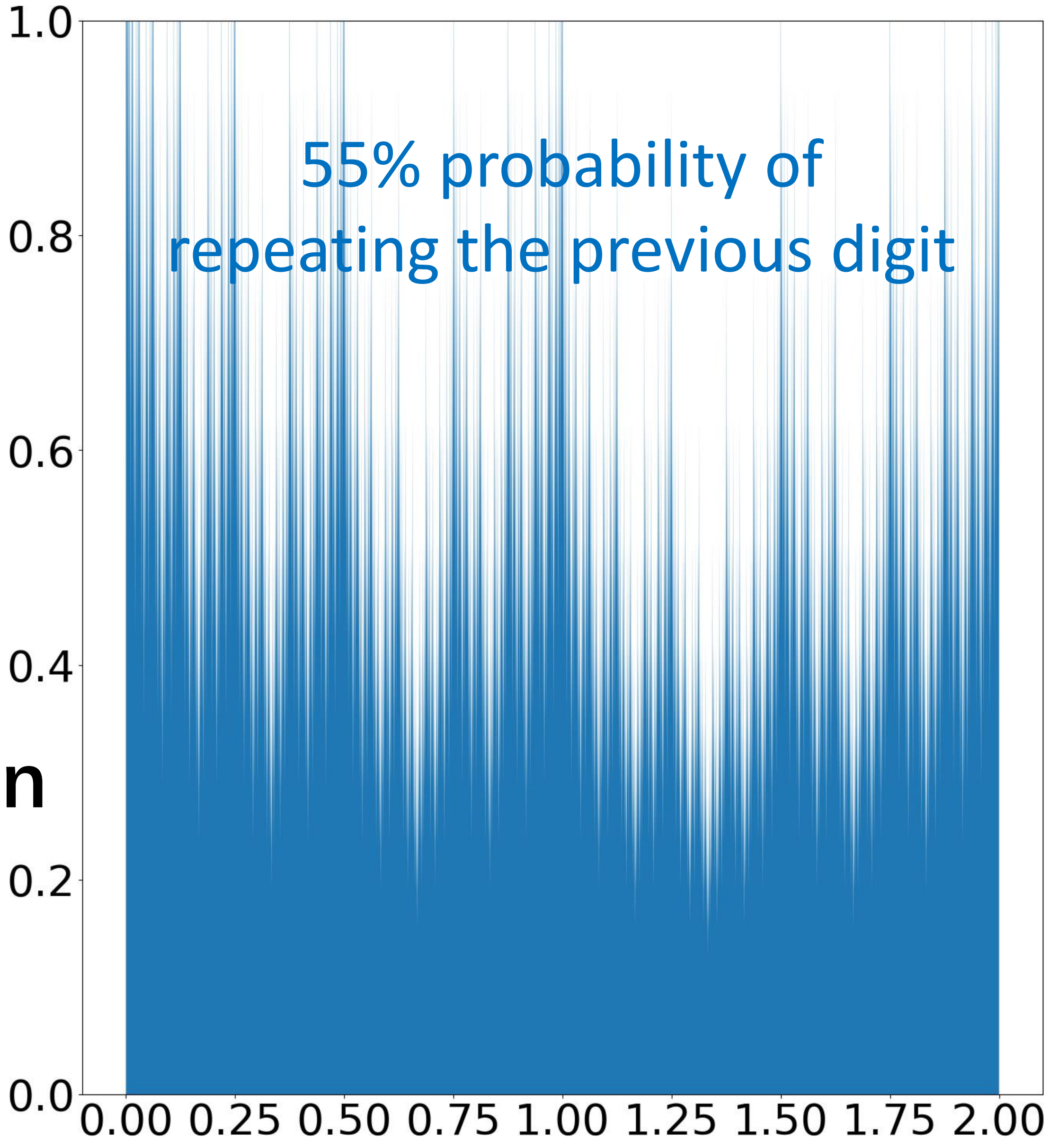
- Physical solution: $x_i^{(k)}$ i.i.d
- Periodic: $x_i^{(k)}$ determined by previous digits
- Quasi-physical: $x_i^{(k)}$ depends on previous digits



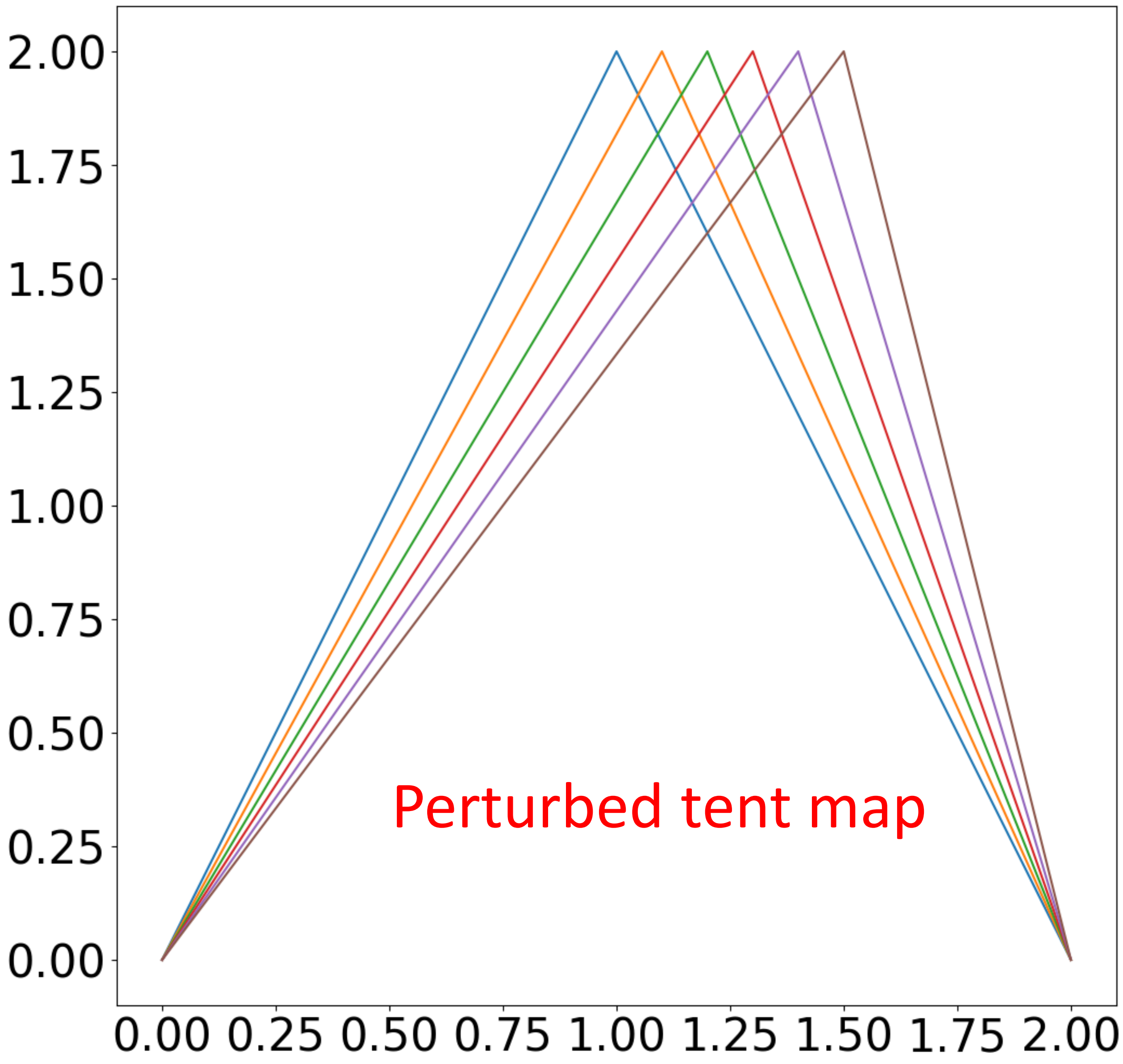
Tent map: Quasi-physical solutions

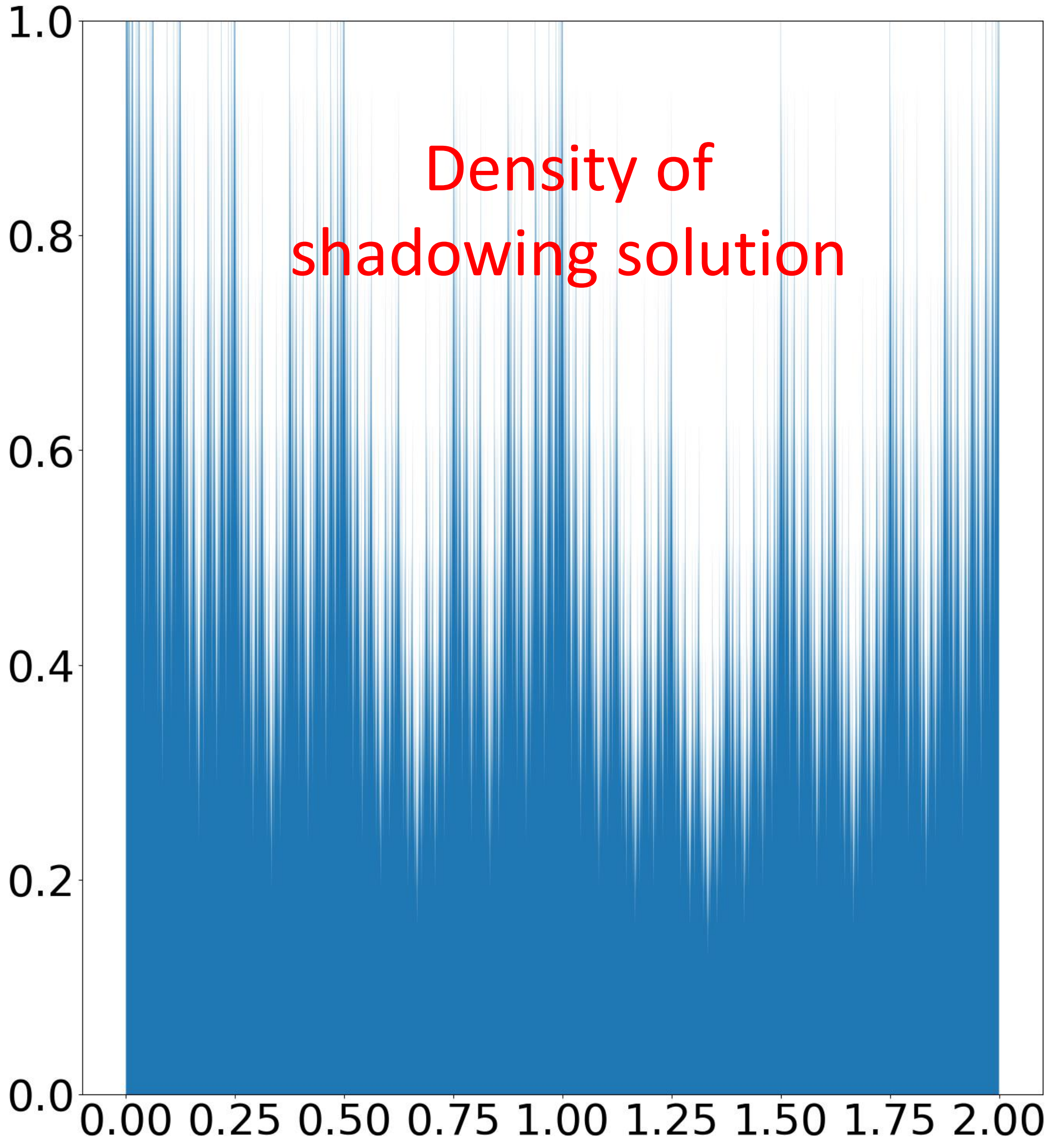
$$x_i = \sum_{k=0}^{\infty} \frac{x_i^{(k)}}{2^k}$$

- Physical solution: $x_i^{(k)}$ i.i.d
- Periodic: $x_i^{(k)}$ determined by previous digits
- Quasi-physical: $x_i^{(k)}$ depends on previous digits

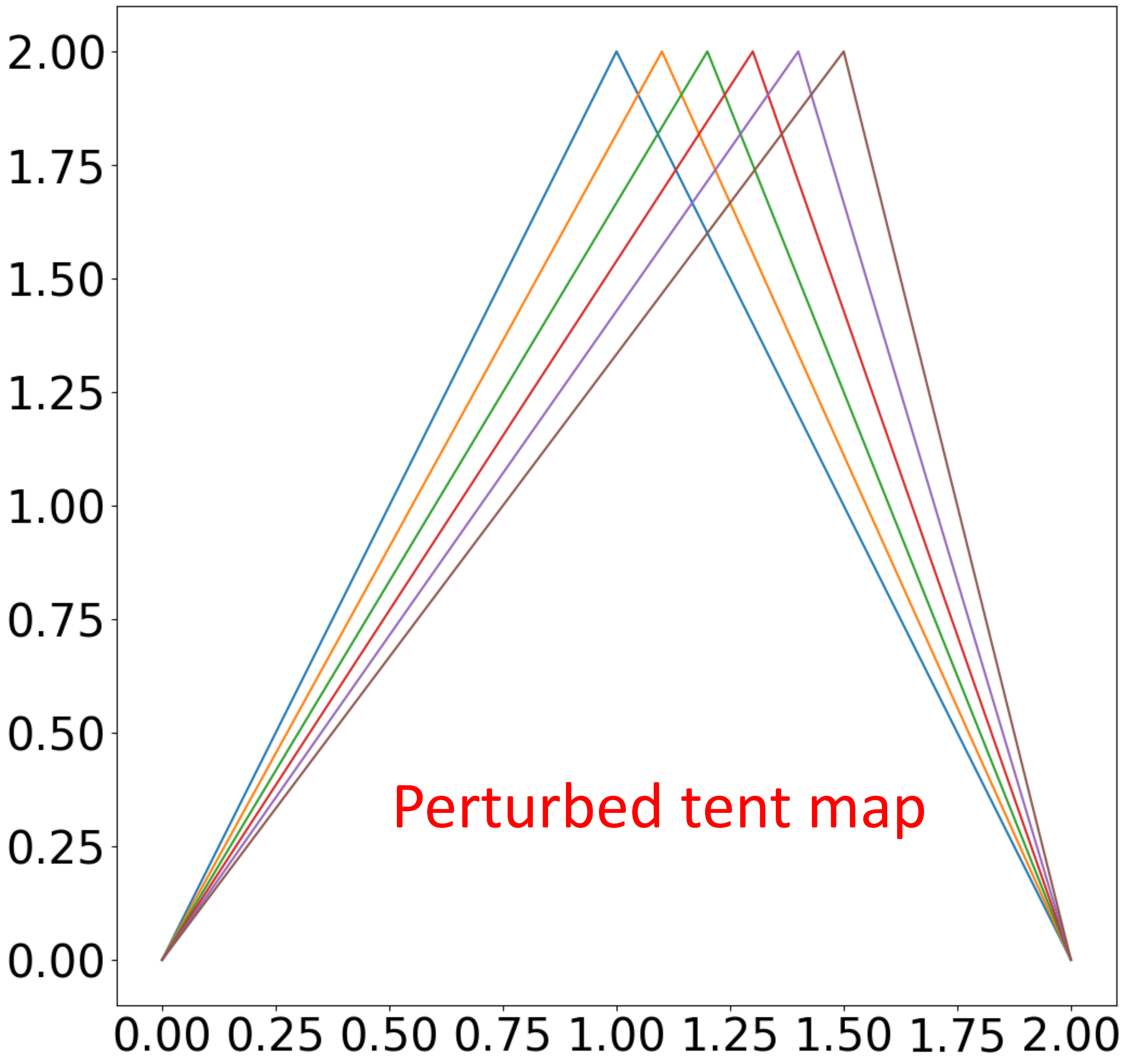


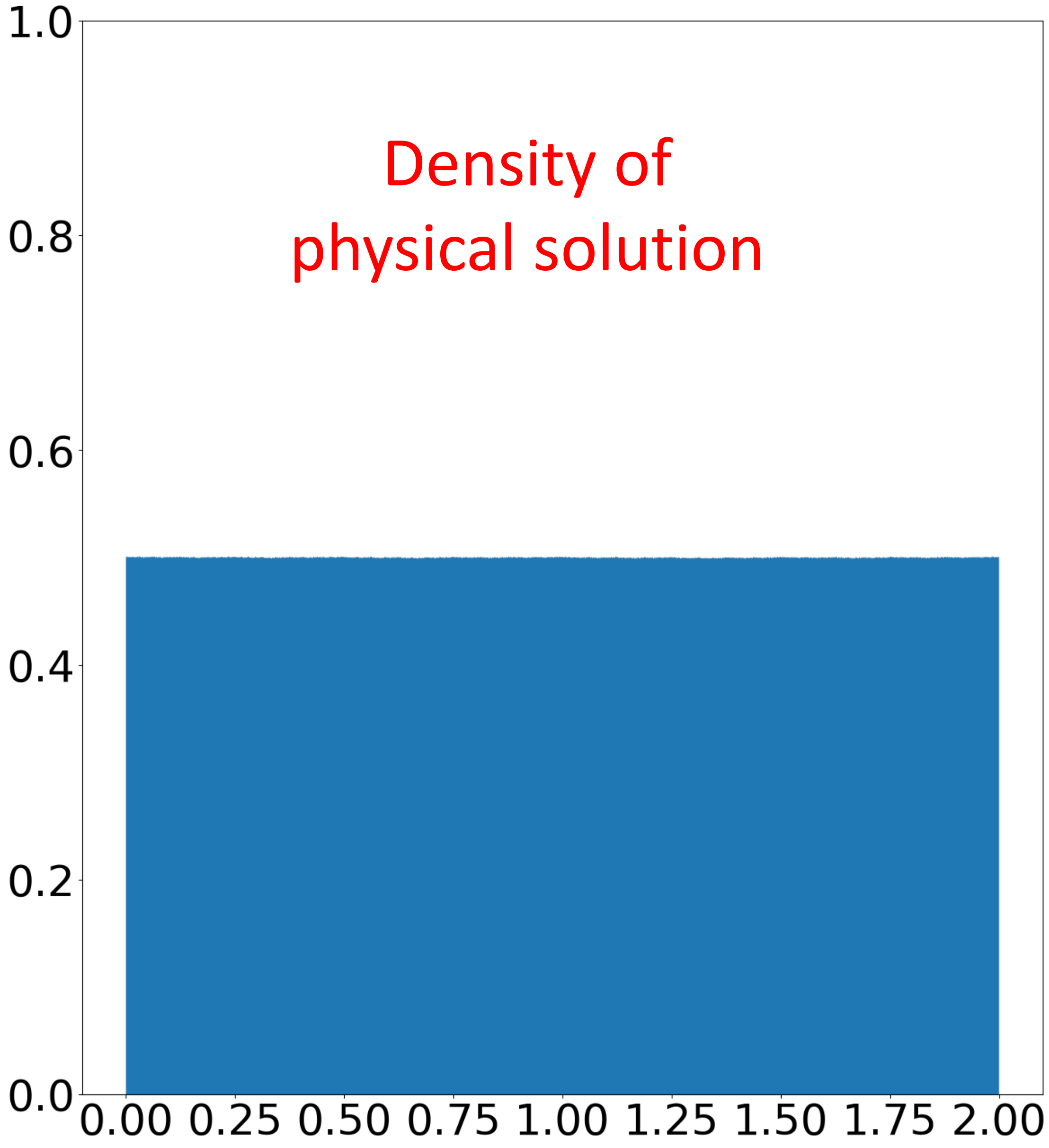
Tent map: Quasi-physical **shadowing** solutions



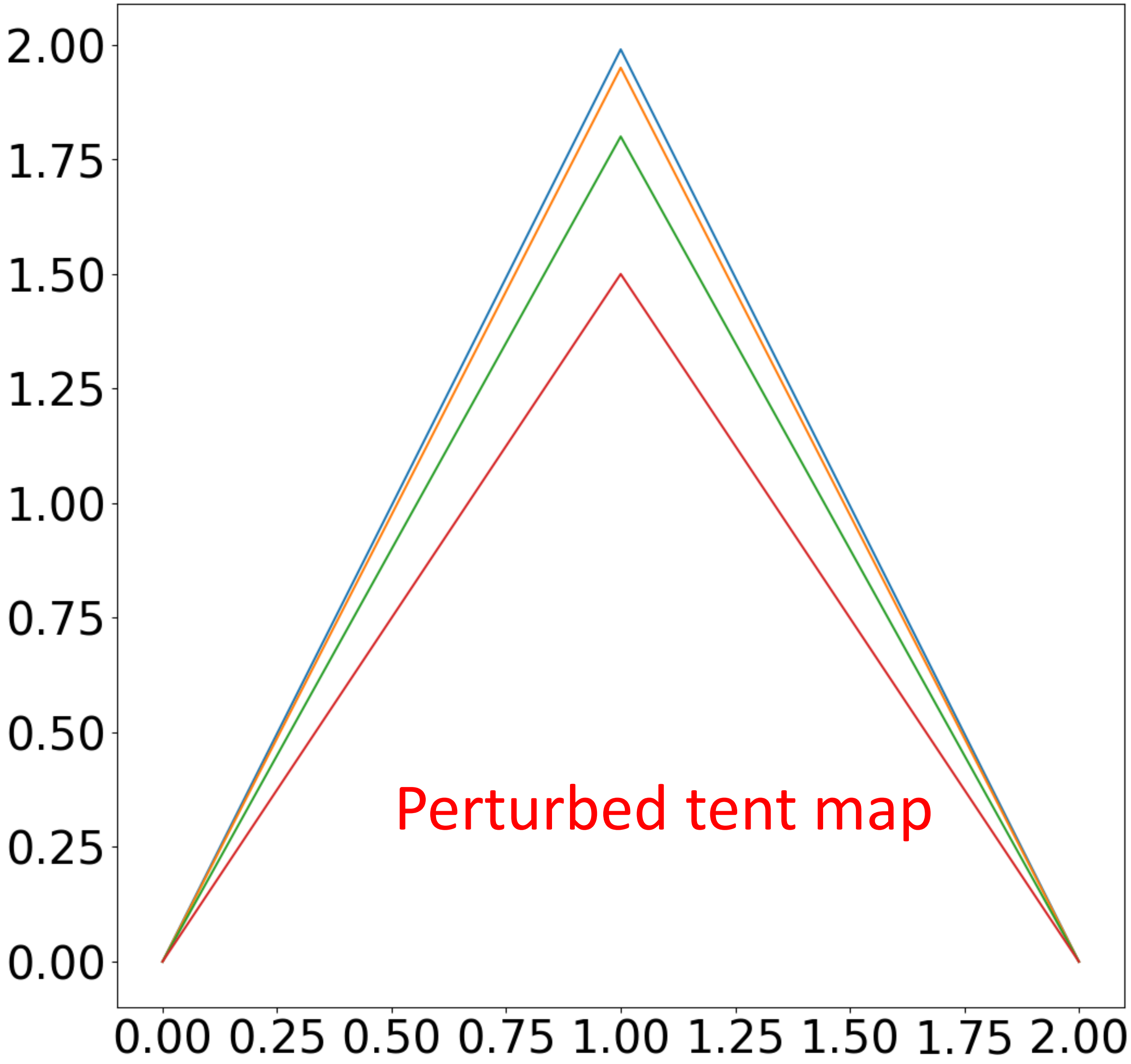


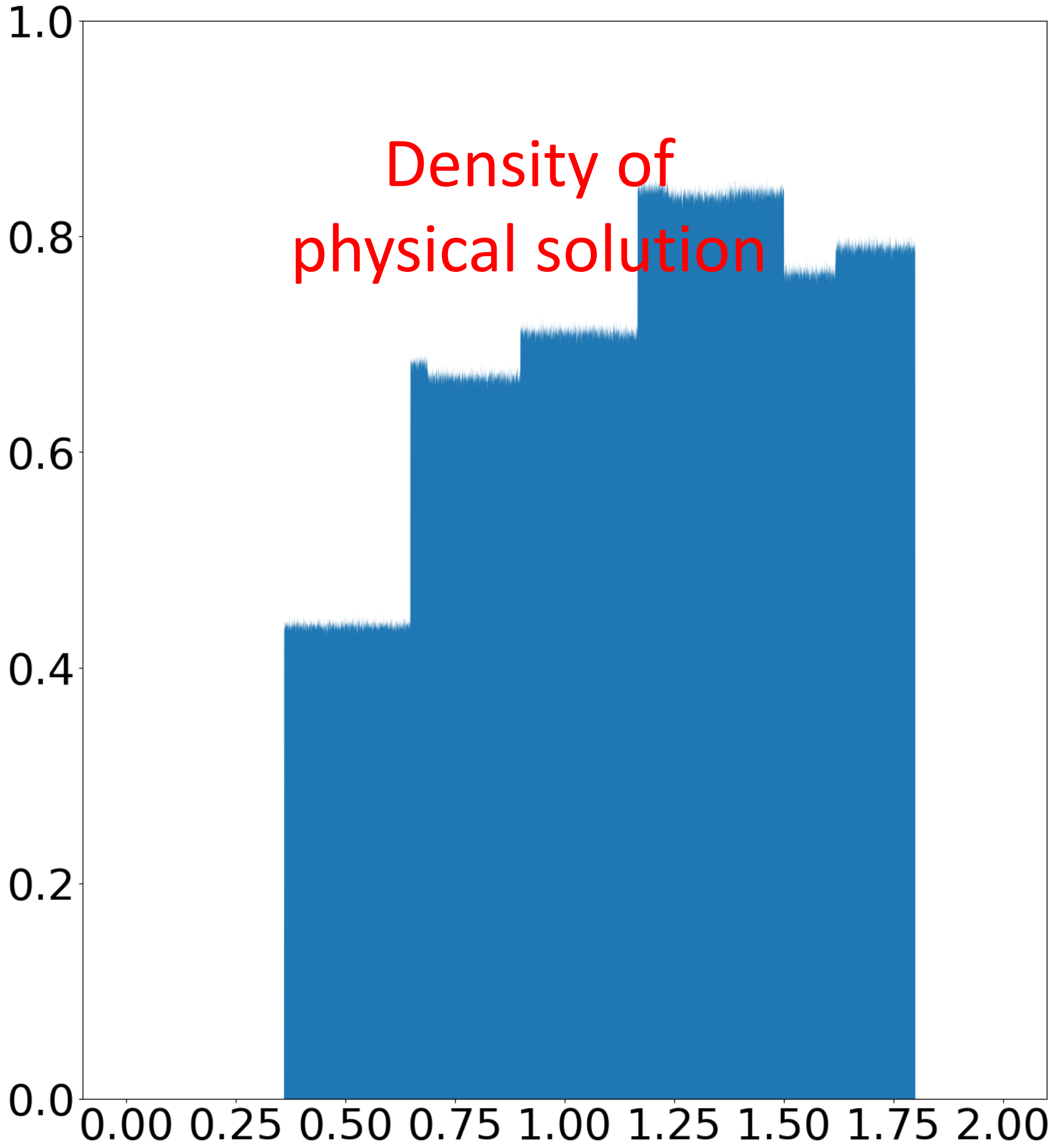
Tent map: Quasi-physical shadowing solutions



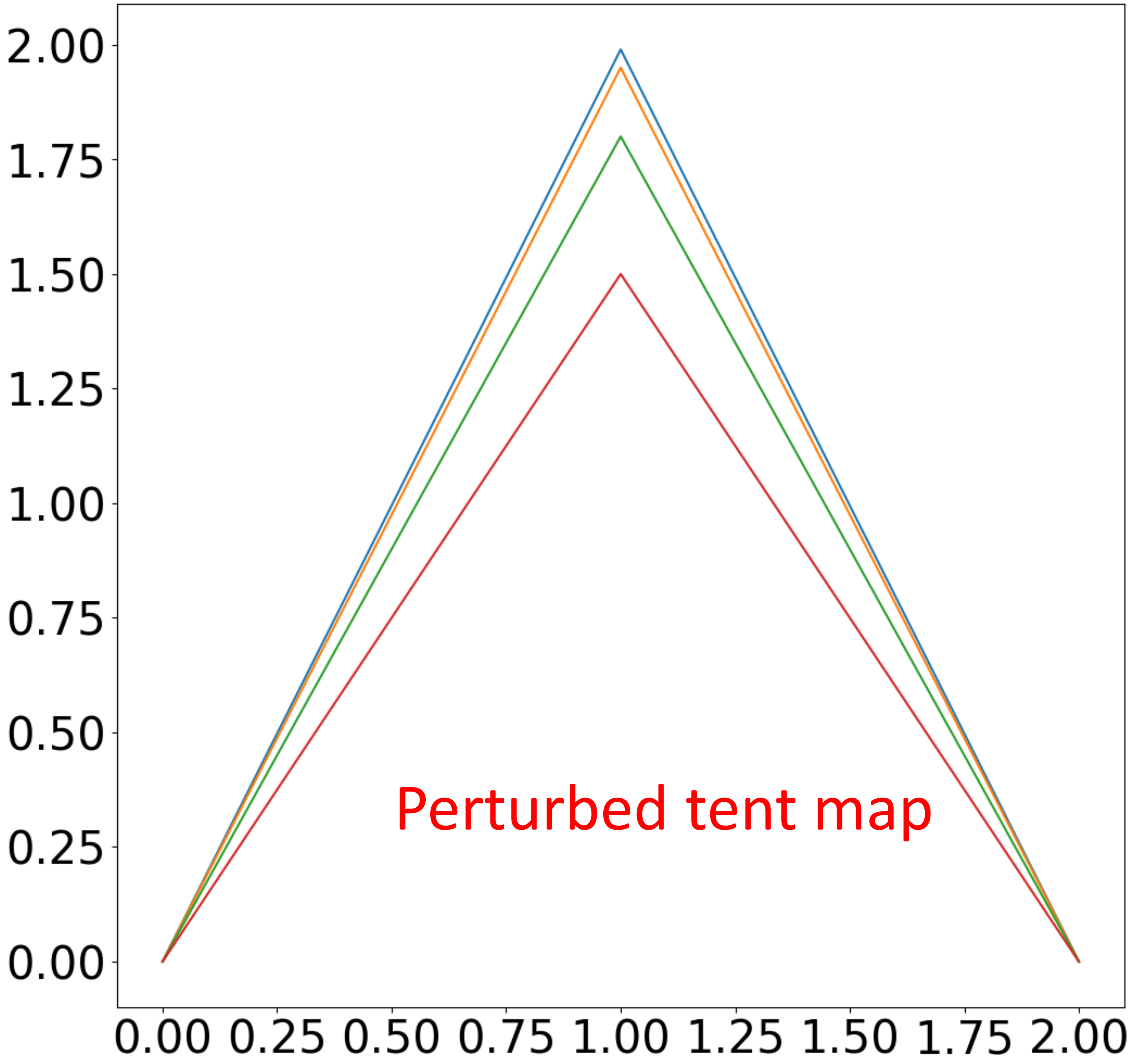


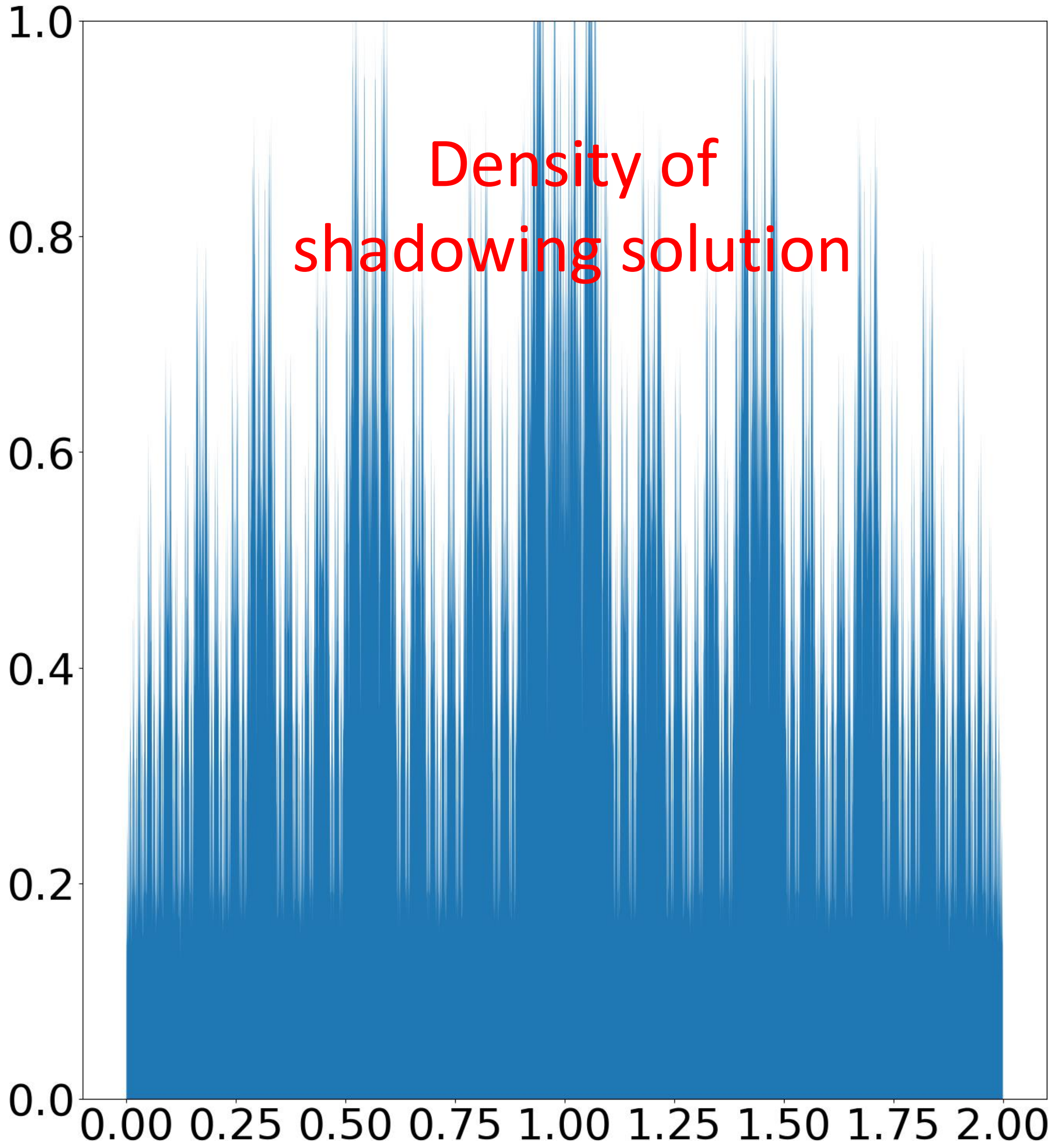
Tent map: Quasi-physical shadowing solutions



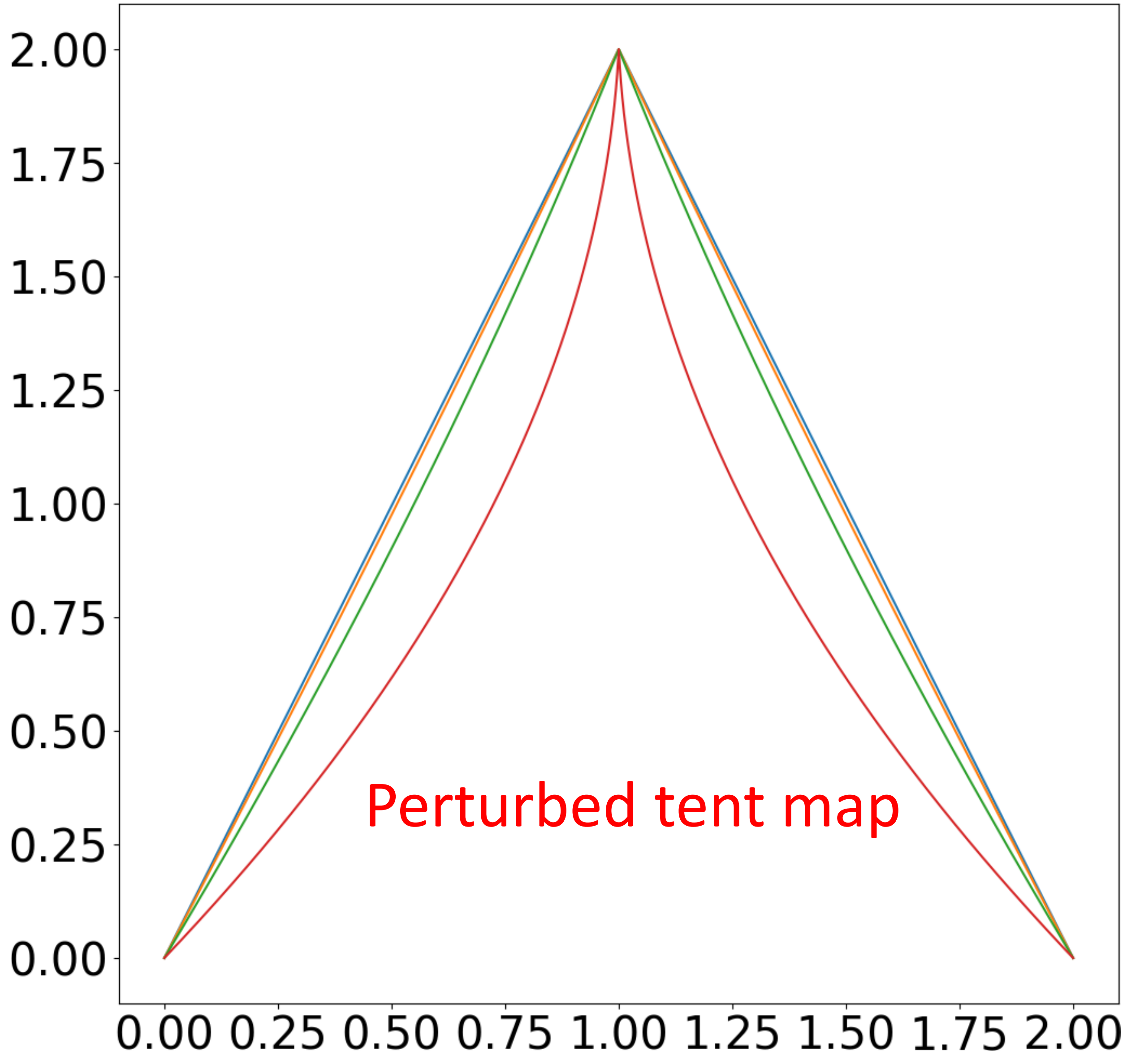


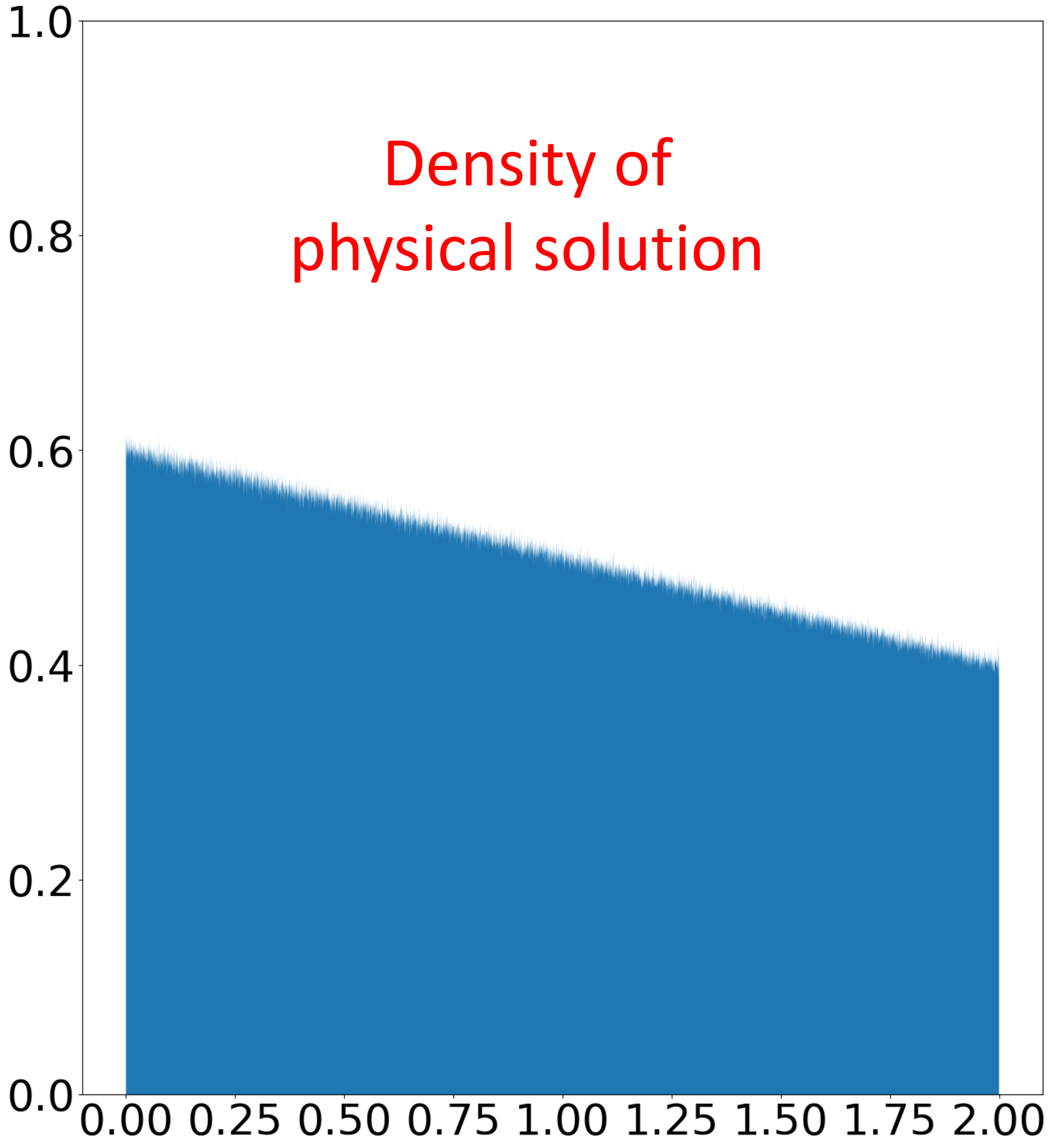
Tent map: Quasi-physical shadowing solutions



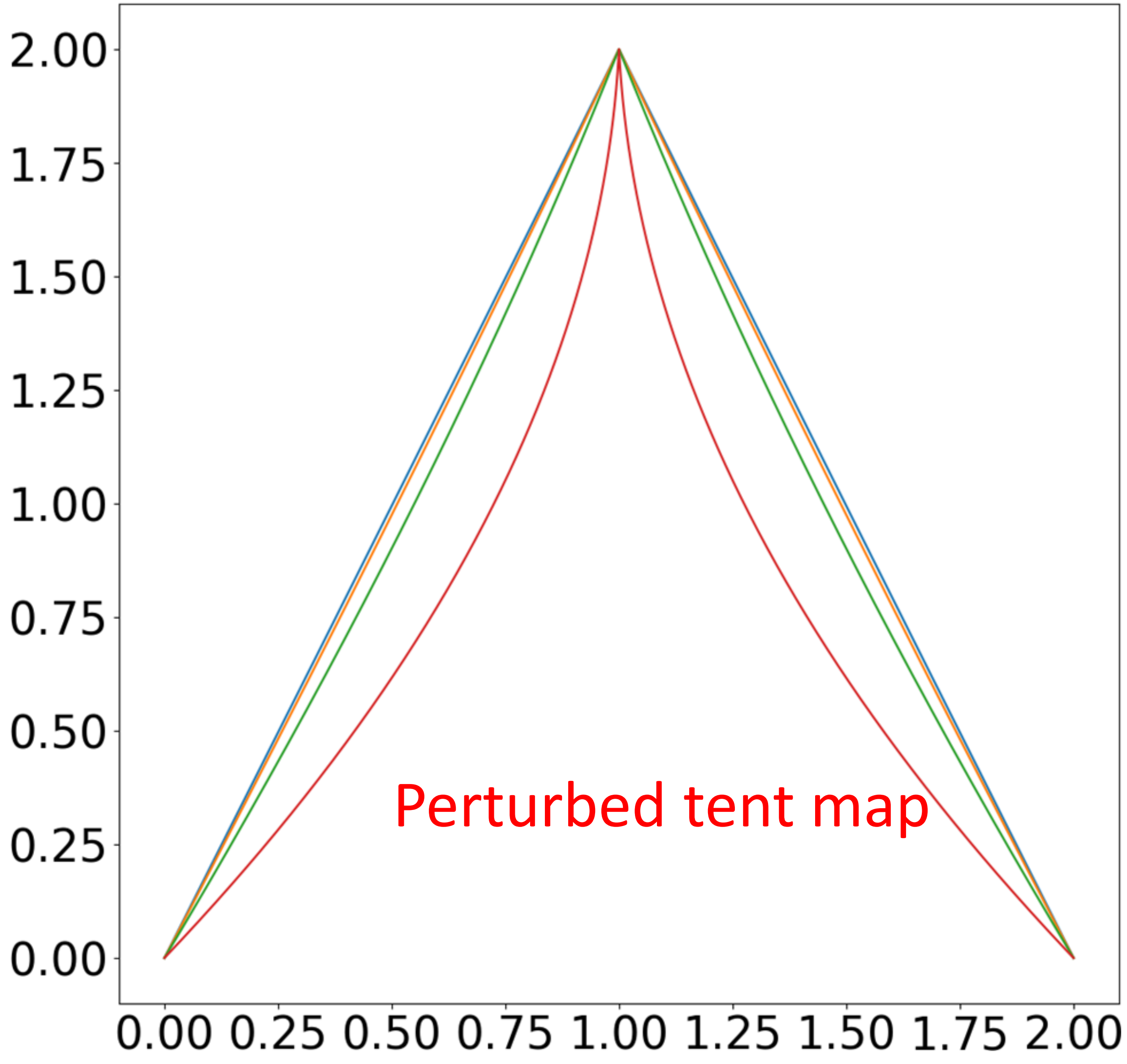


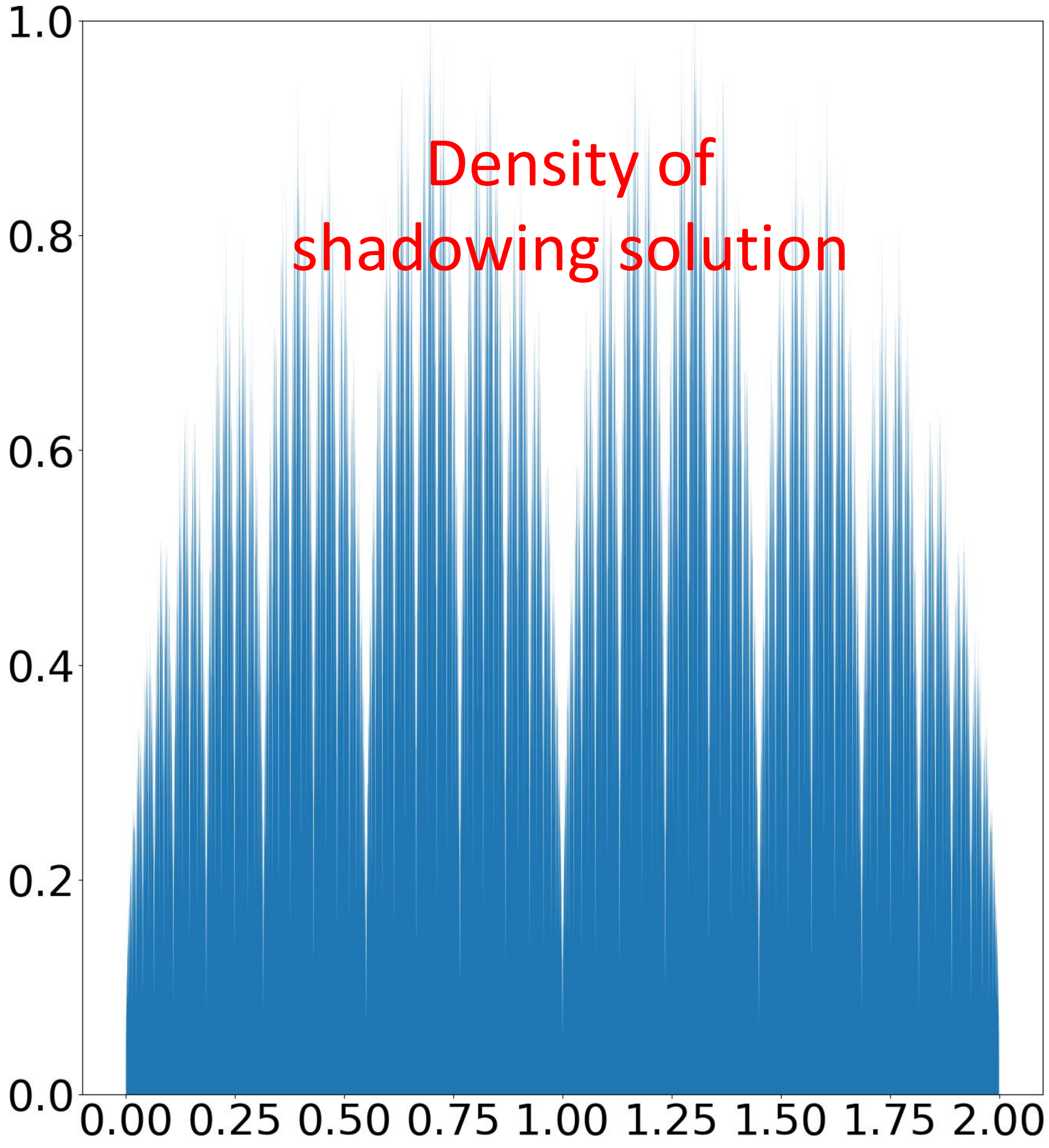
Tent map: Quasi-physical shadowing solutions



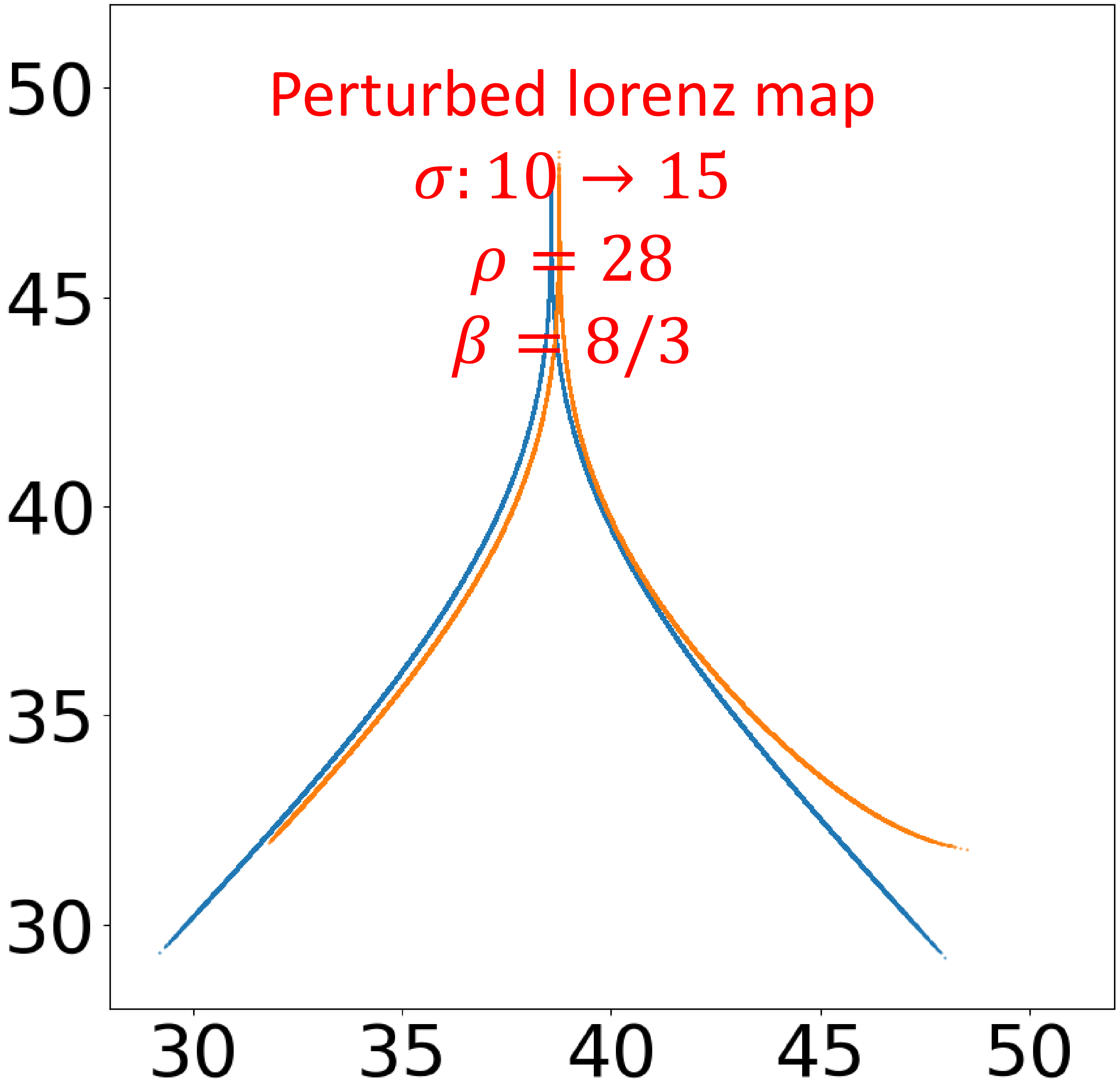


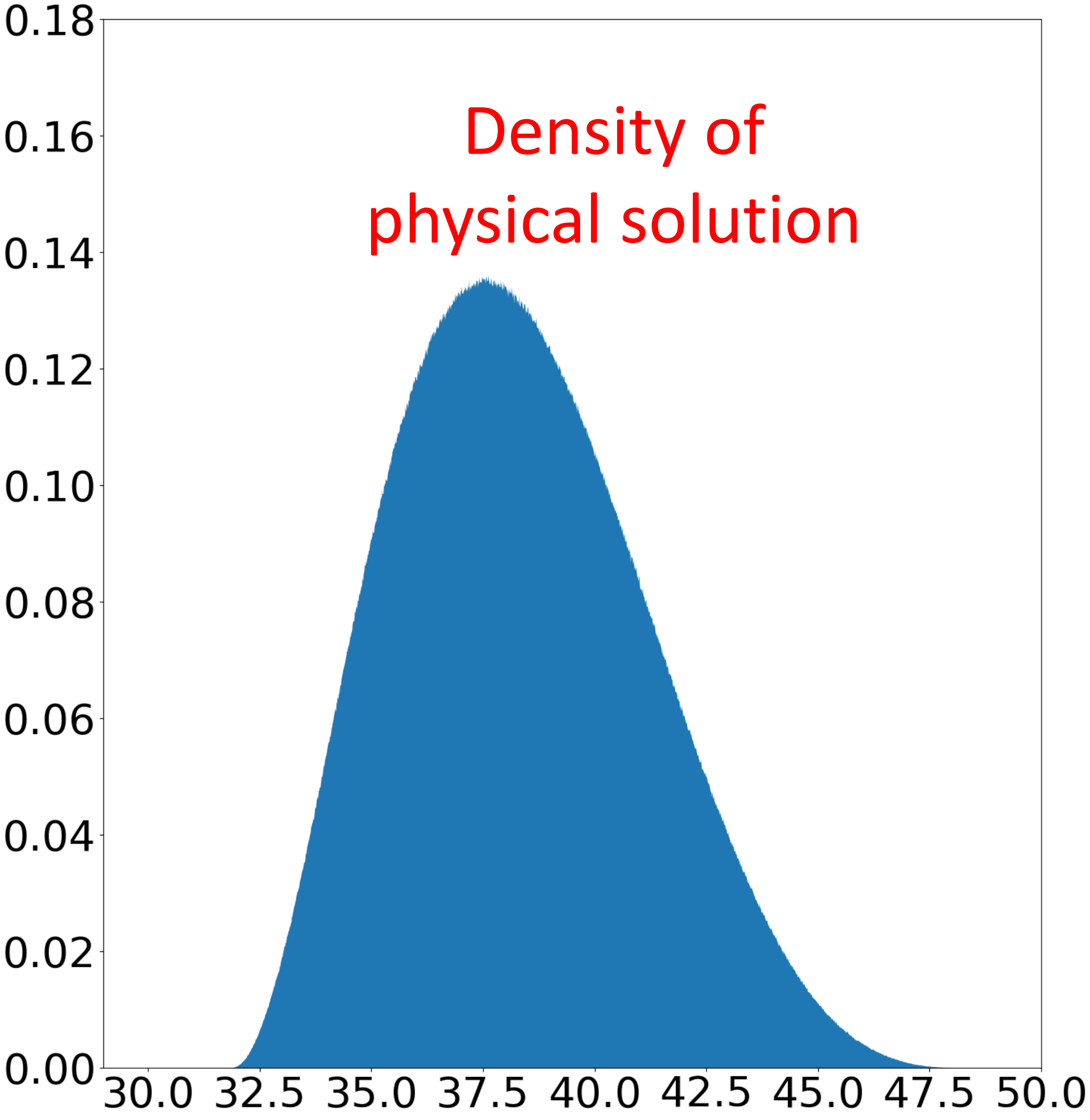
Tent map: Quasi-physical shadowing solutions



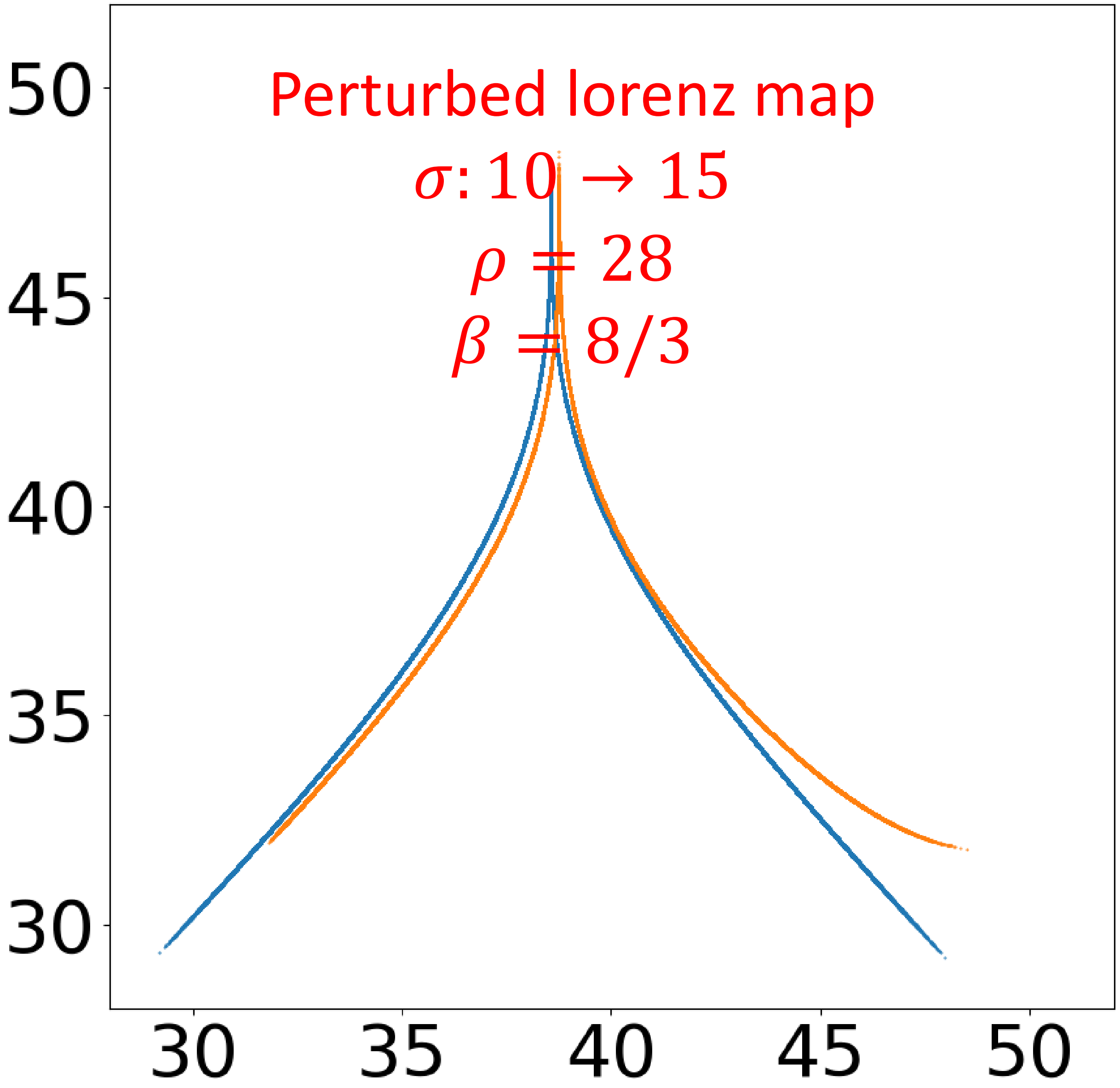


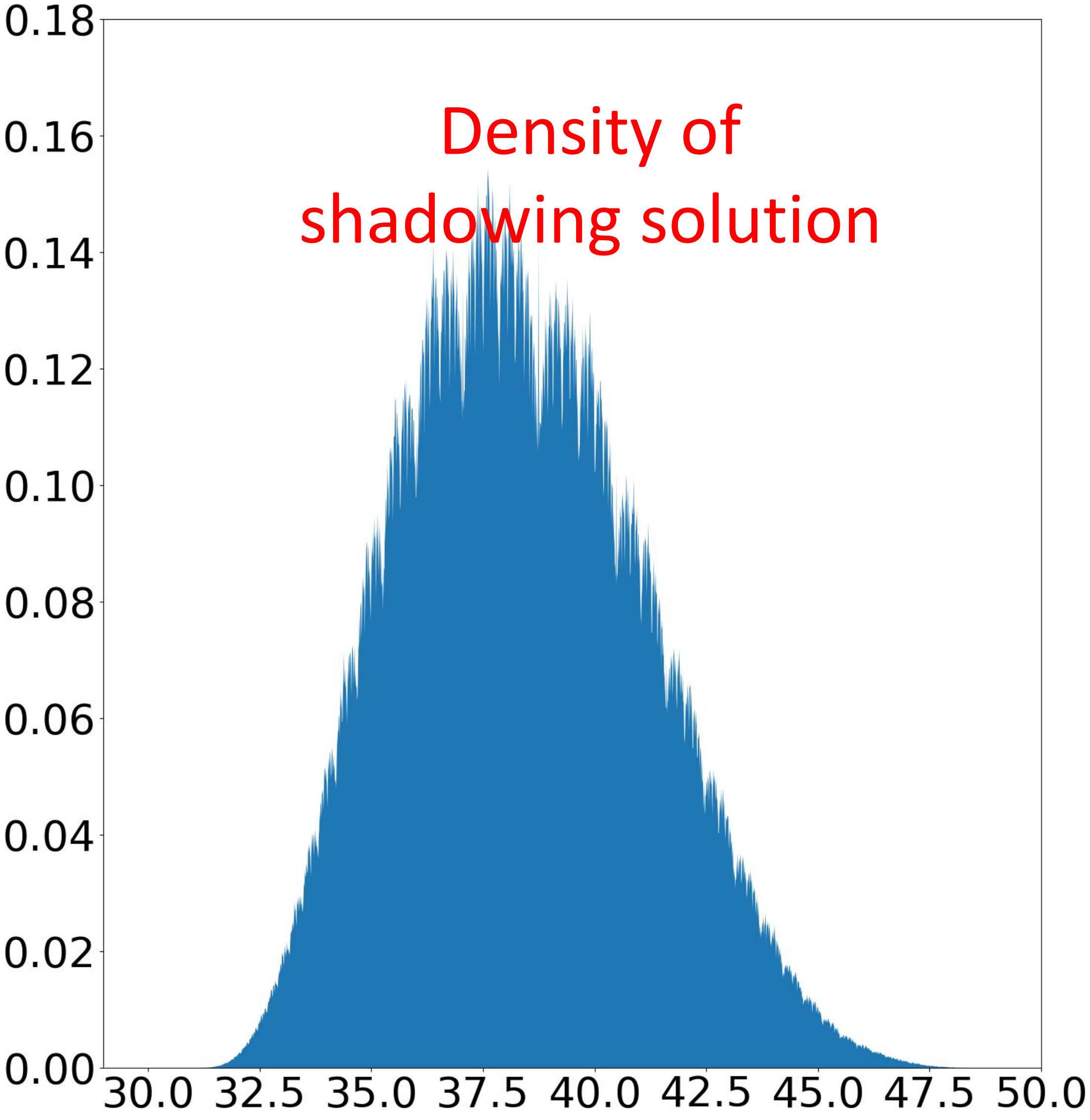
Lorenz map: Quasi-physical shadowing solutions



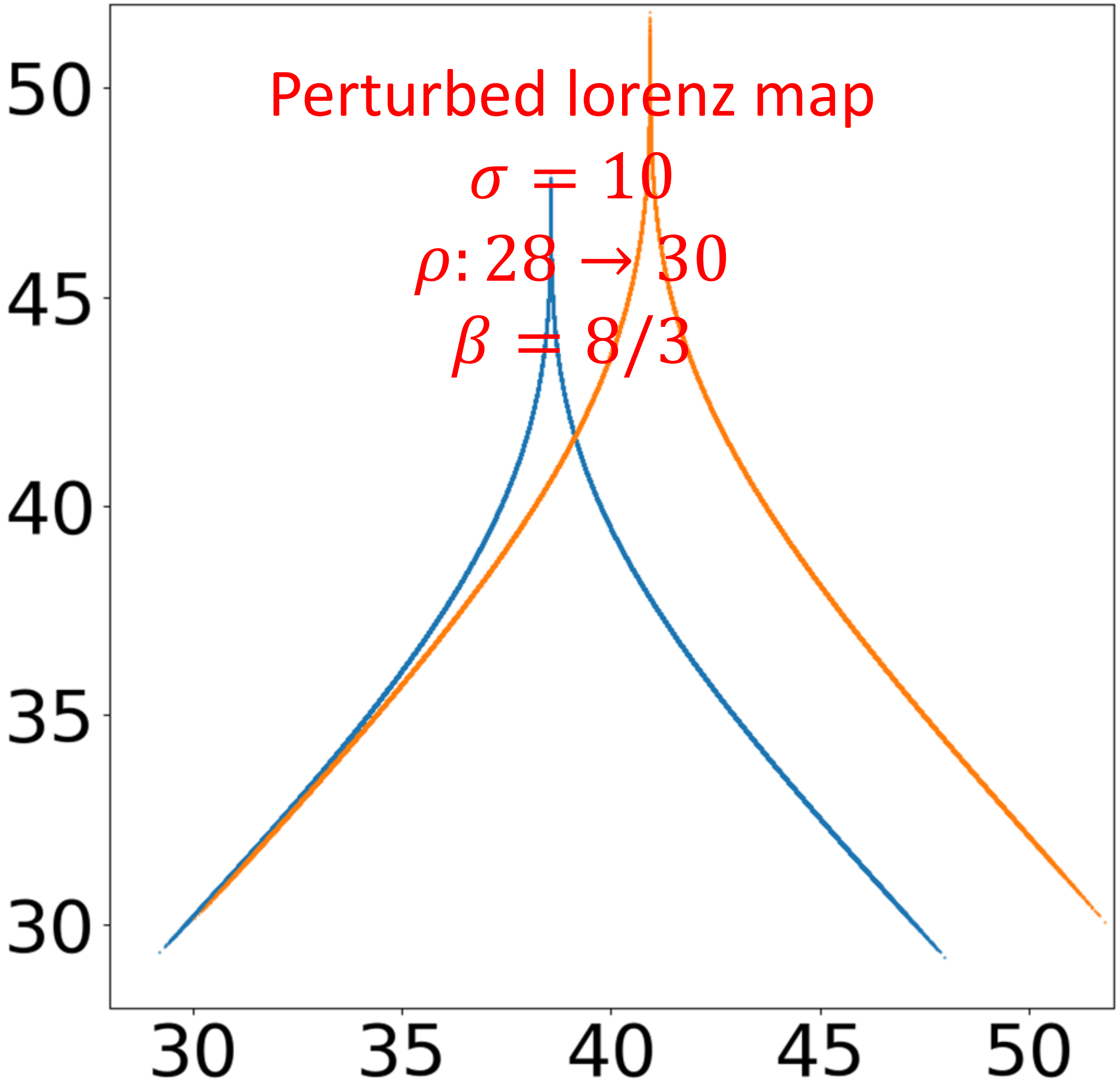


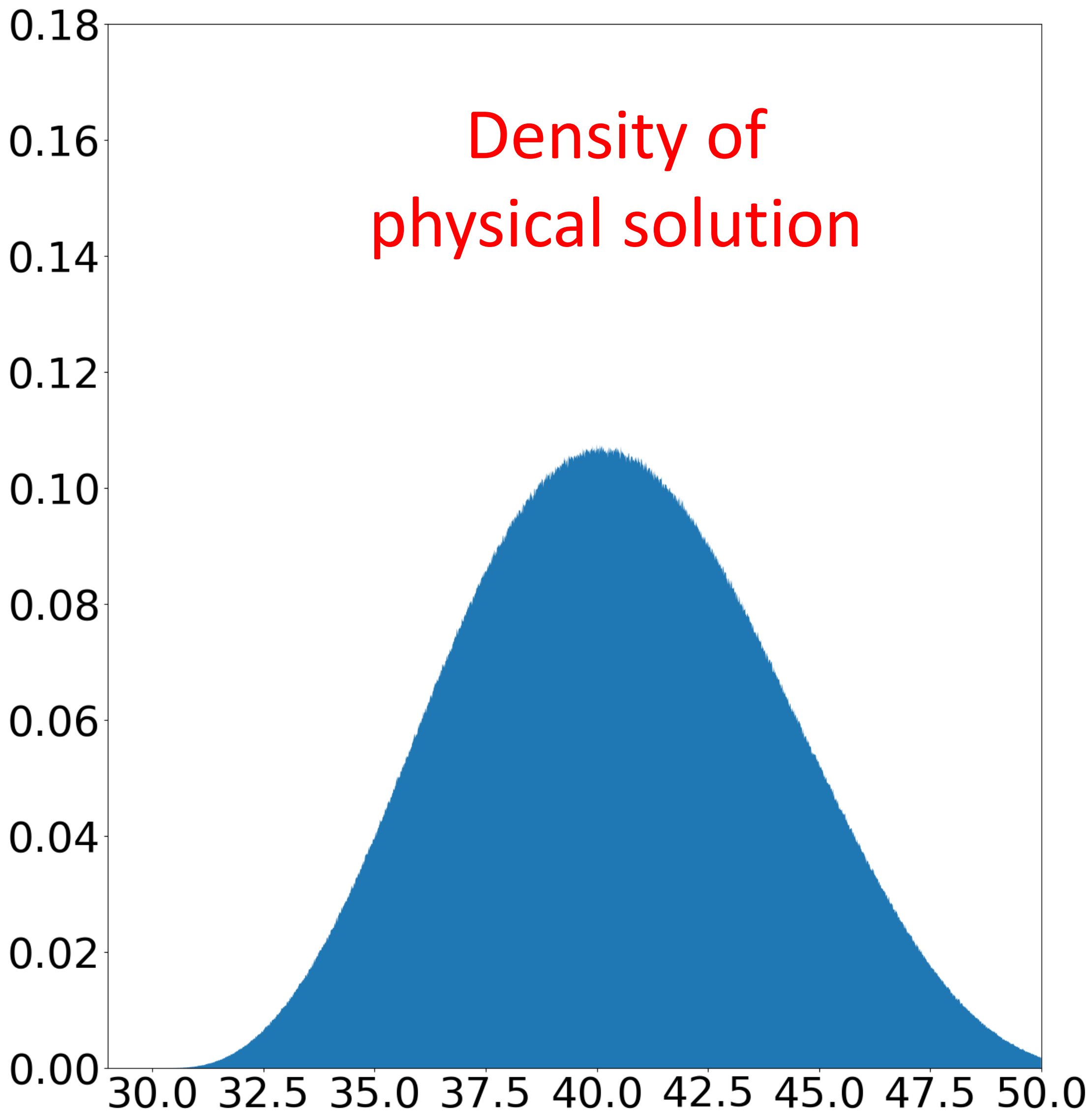
Lorenz map: Quasi-physical shadowing solutions



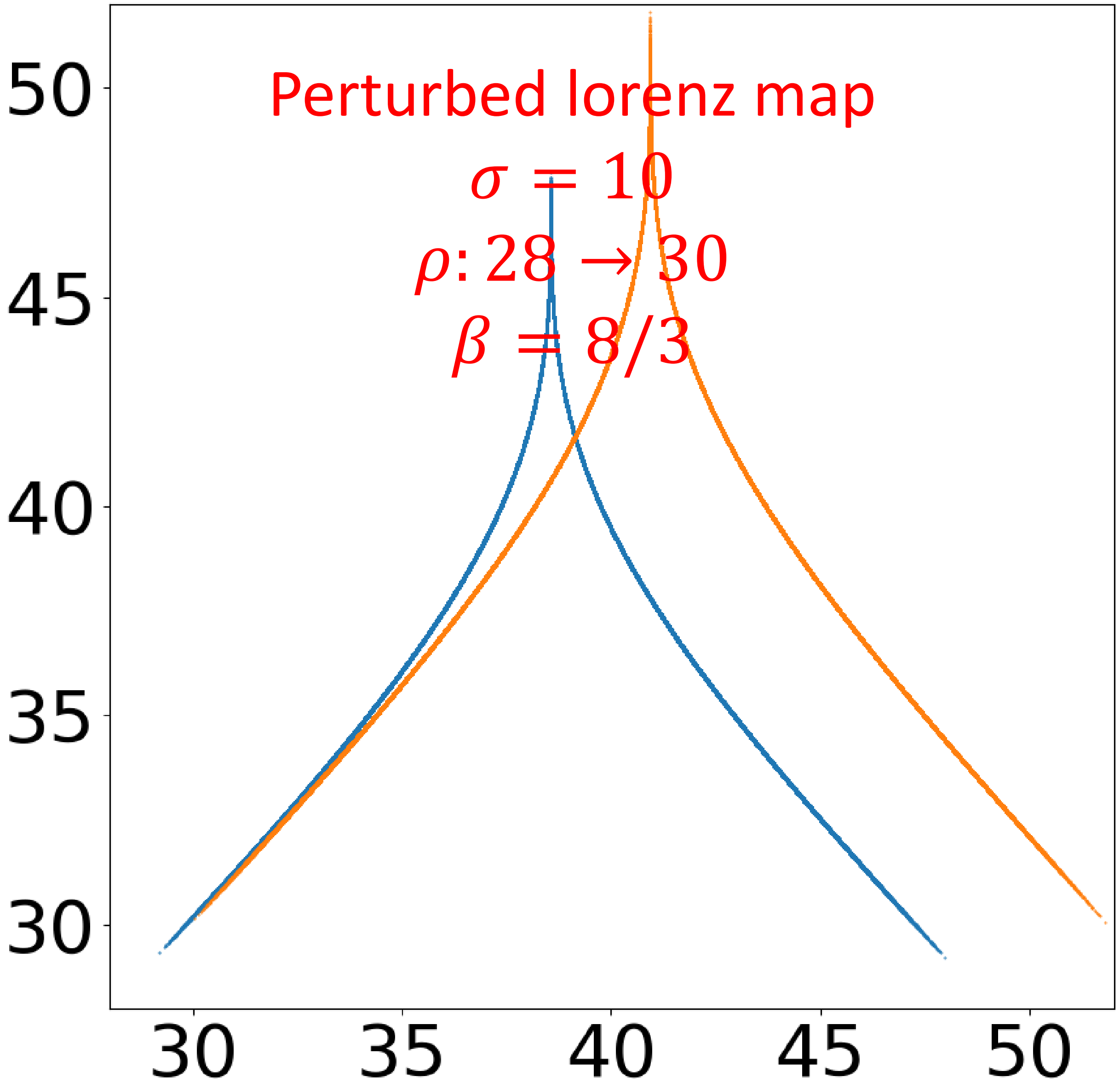


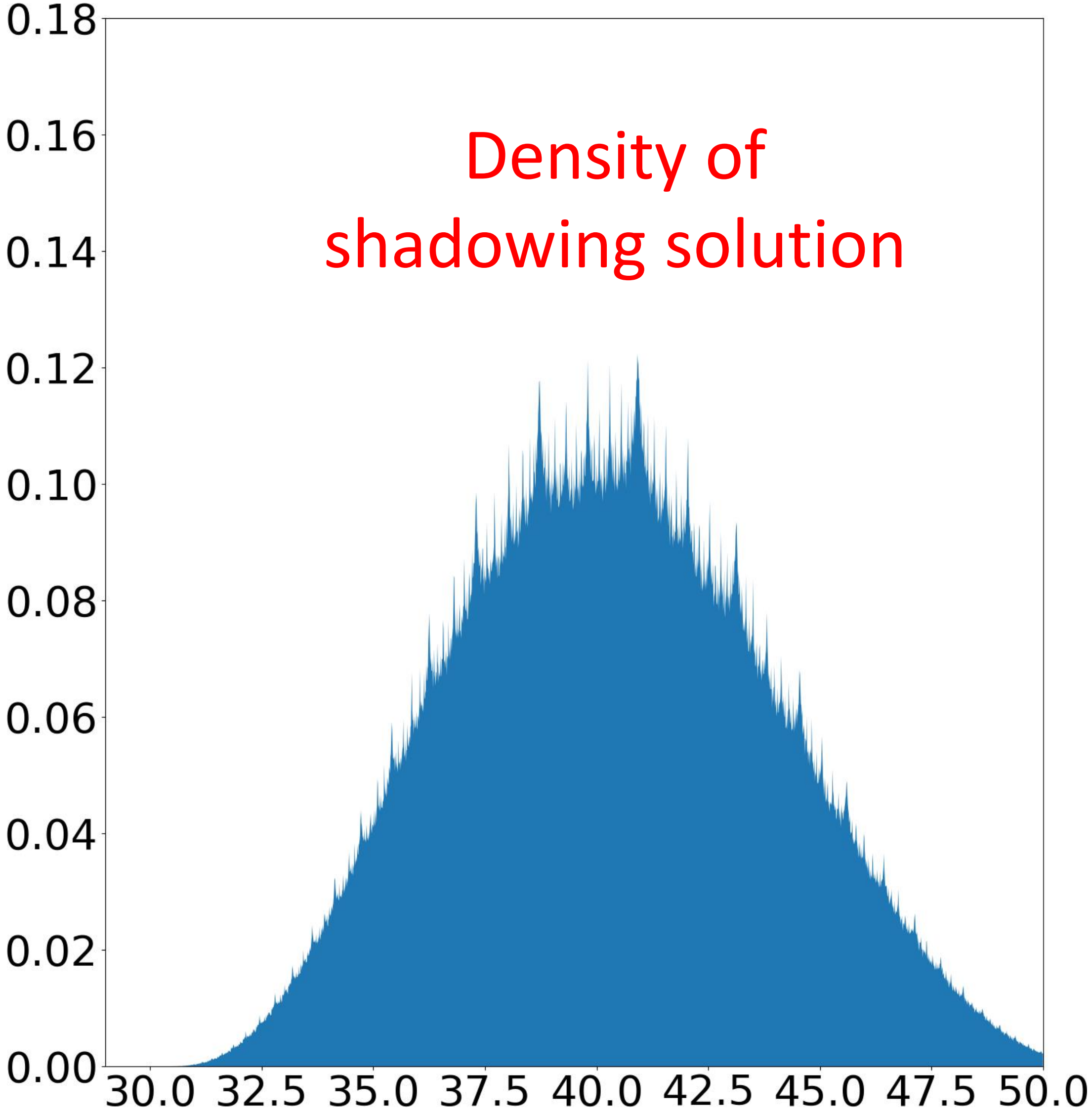
Lorenz map: Quasi-physical shadowing solutions



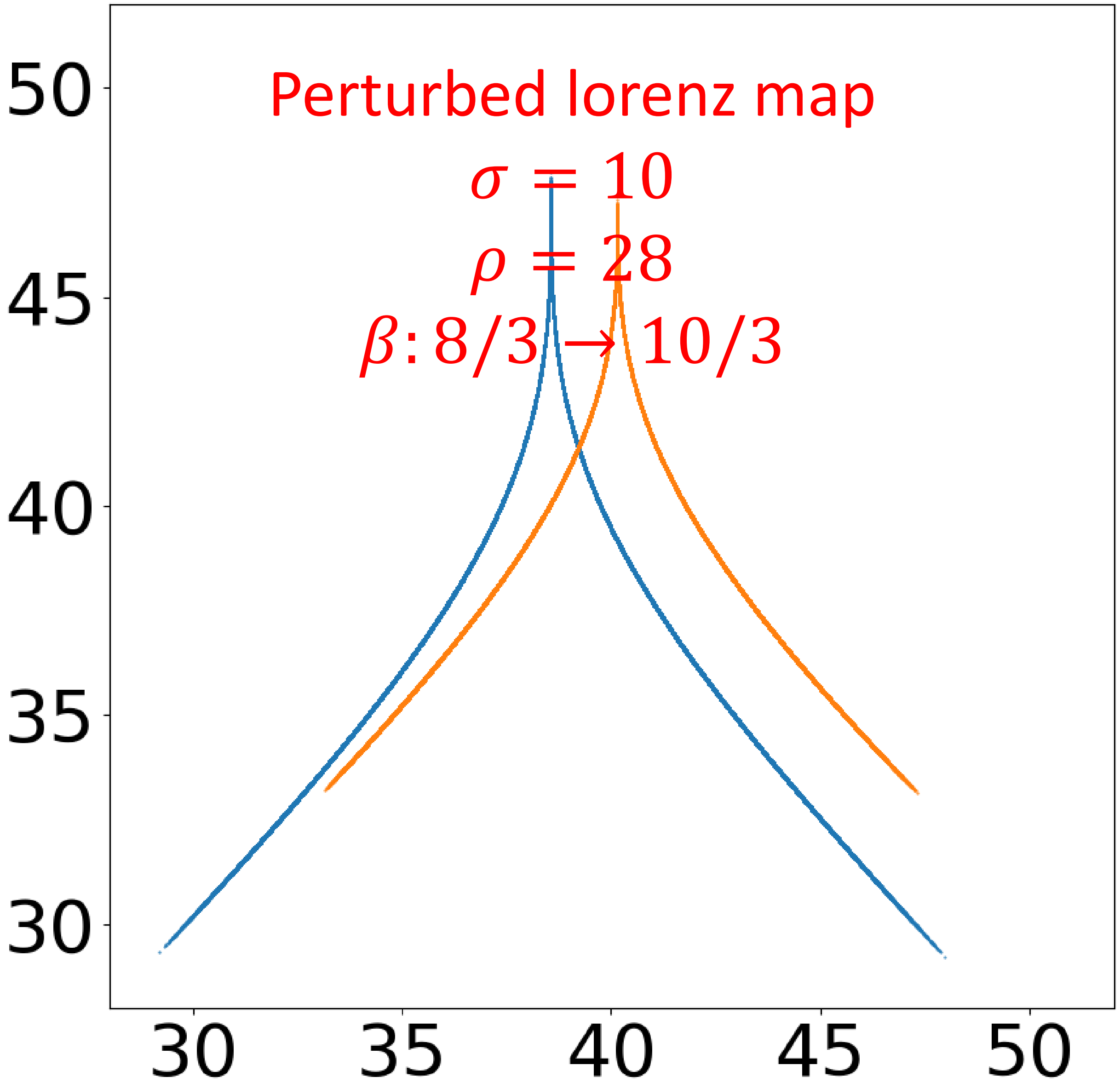


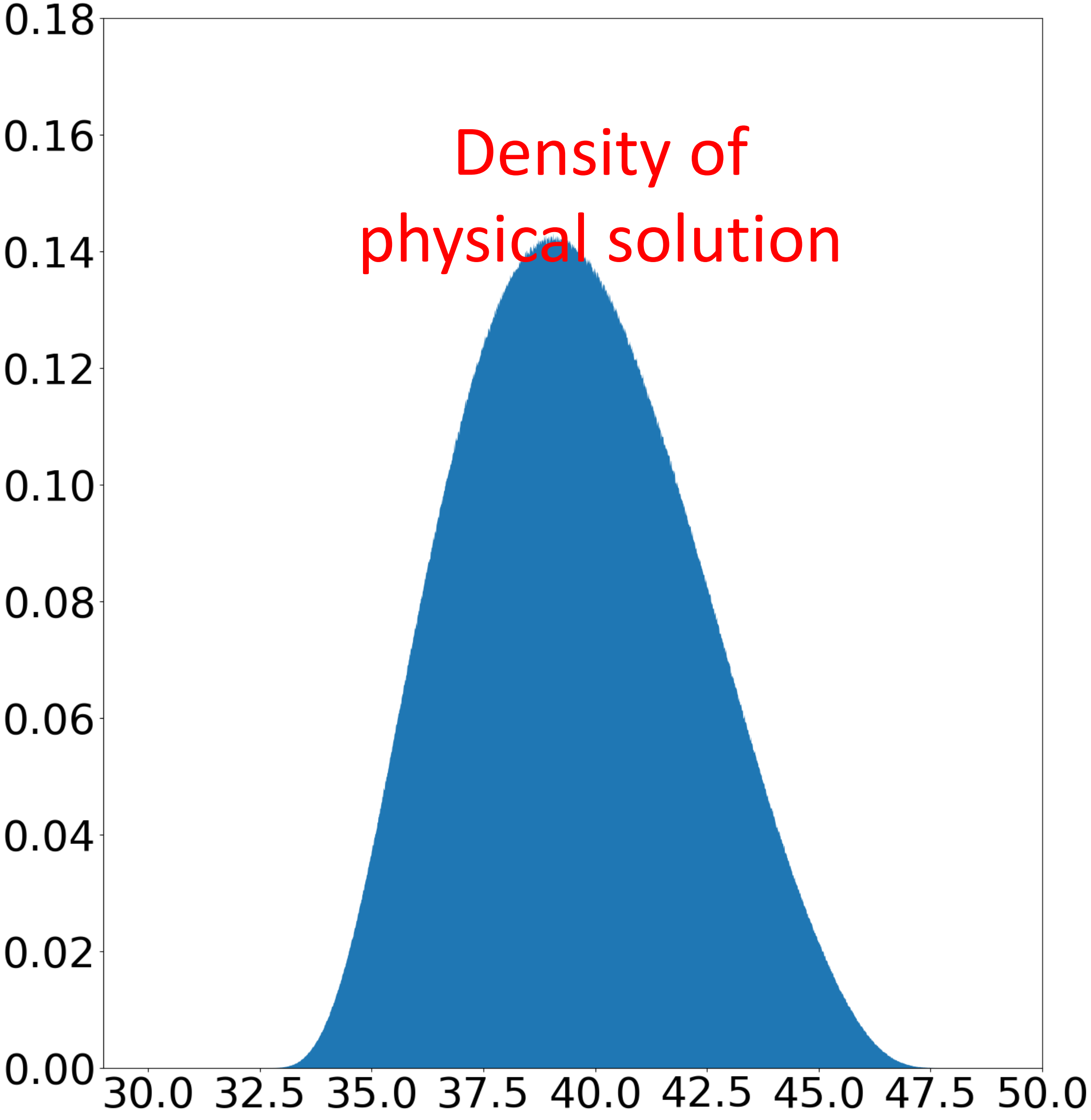
Lorenz map: Quasi-physical shadowing solutions



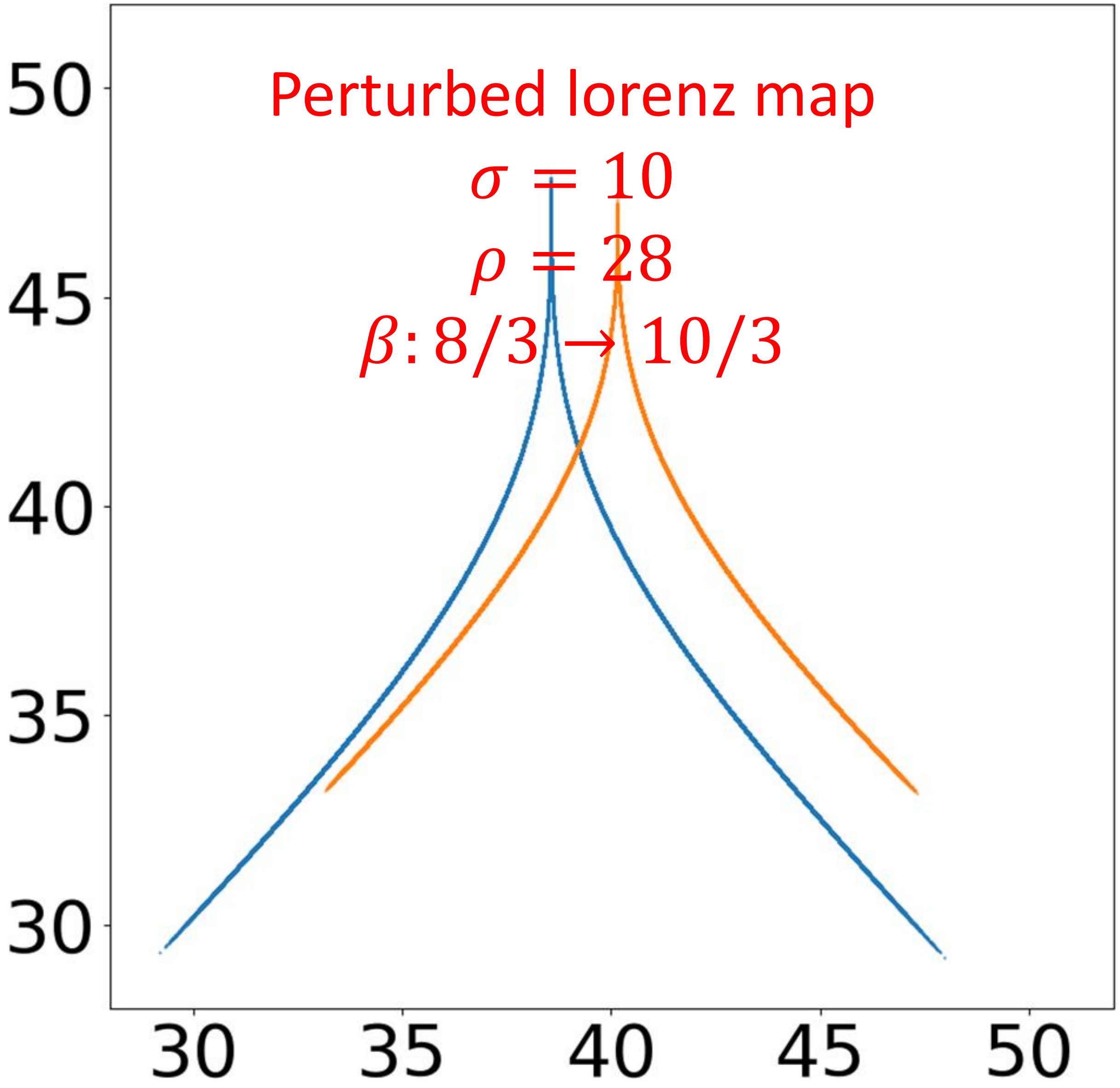


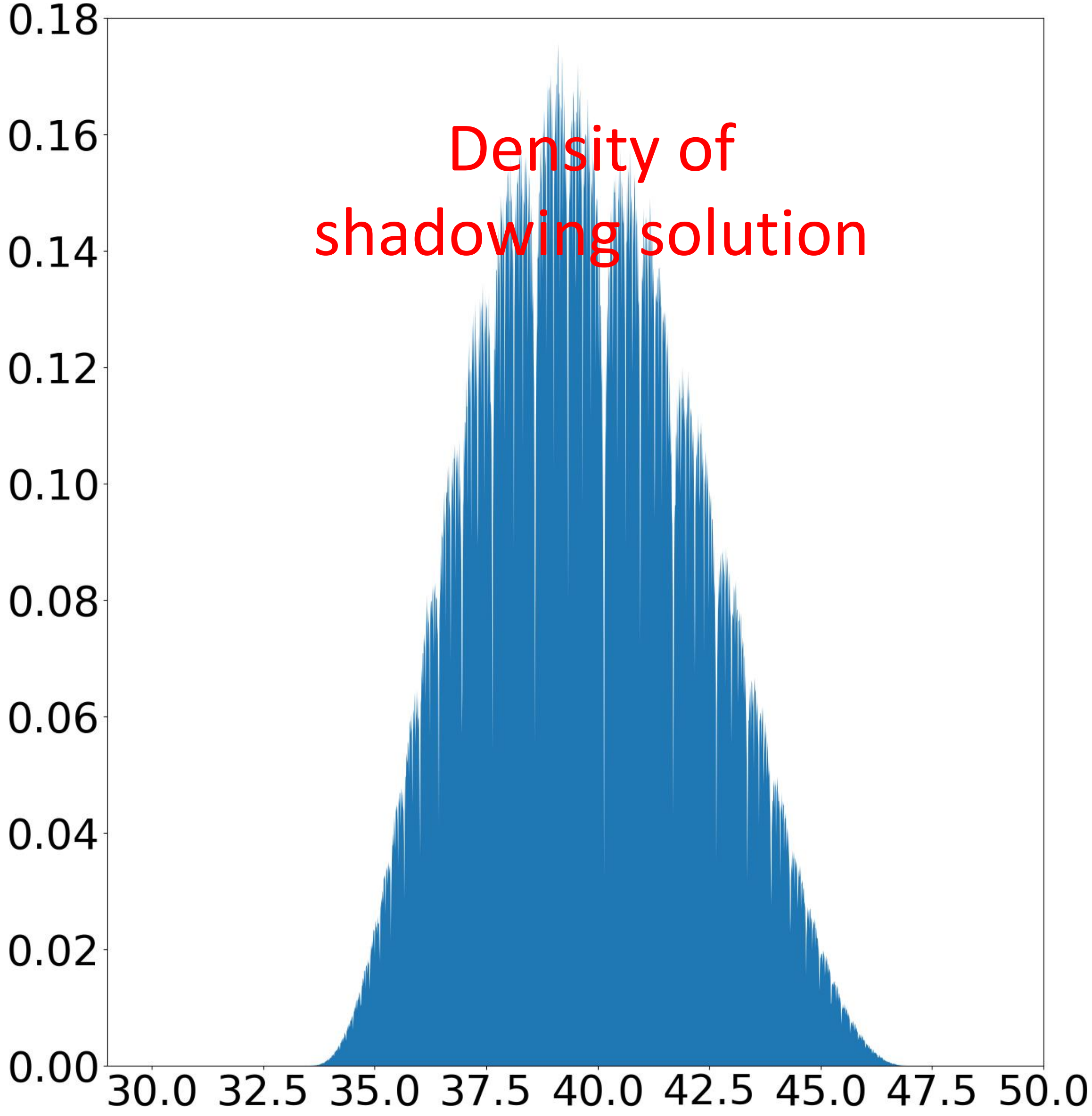
Lorenz map: Quasi-physical shadowing solutions





Lorenz map: Quasi-physical shadowing solutions





Can a butterfly control the climate?

NO

- Ergodicity
- Shadowing

YES

- Ergodicity not applicable
- **Shadowing nonphysical**
- **Evidence in model system**

Can a **butterfly control the climate**?

NO

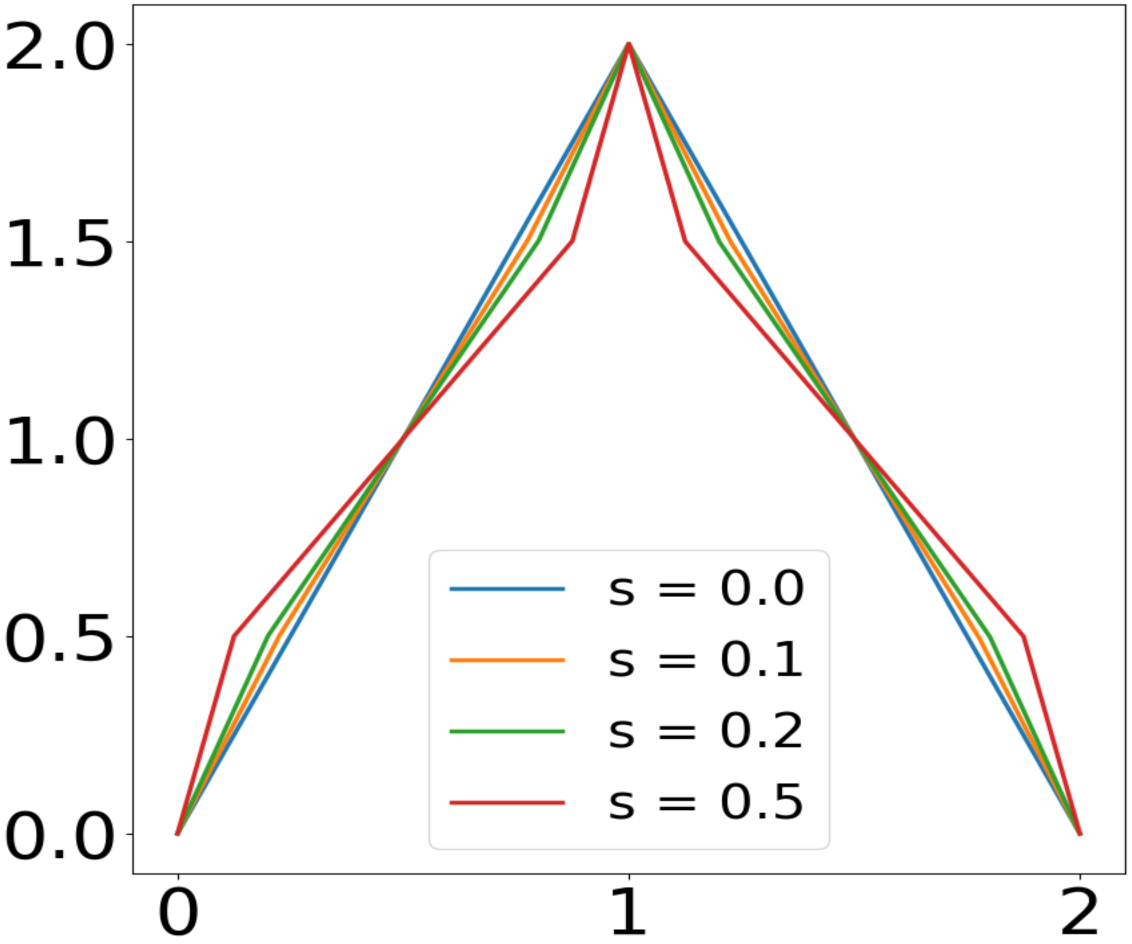
- Ergodicity
- Shadowing

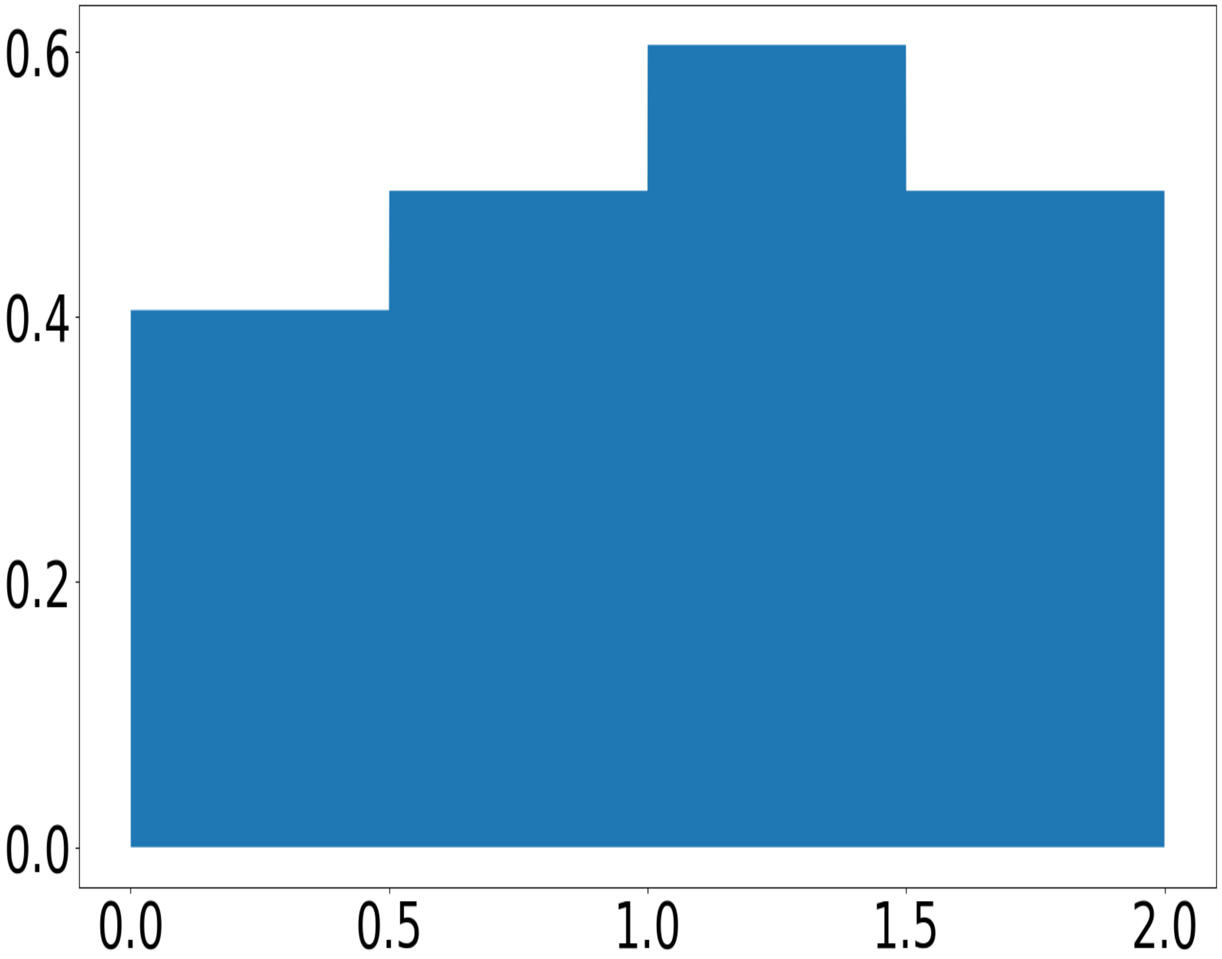
YES

- Ergodicity not applicable
- Shadowing nonphysical
- **Evidence in model system**

Can a butterfly control the climate?

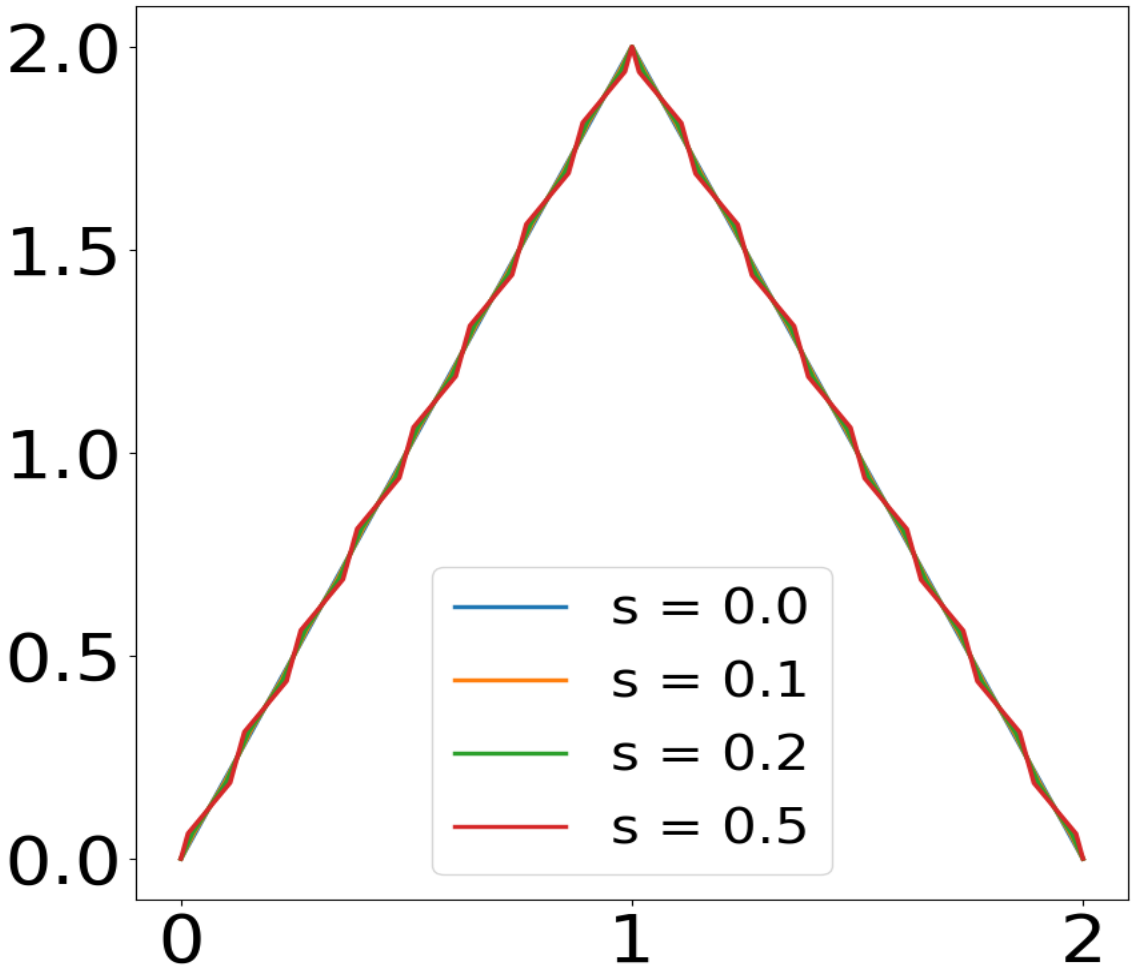
Evidence in model systems – large perturbation

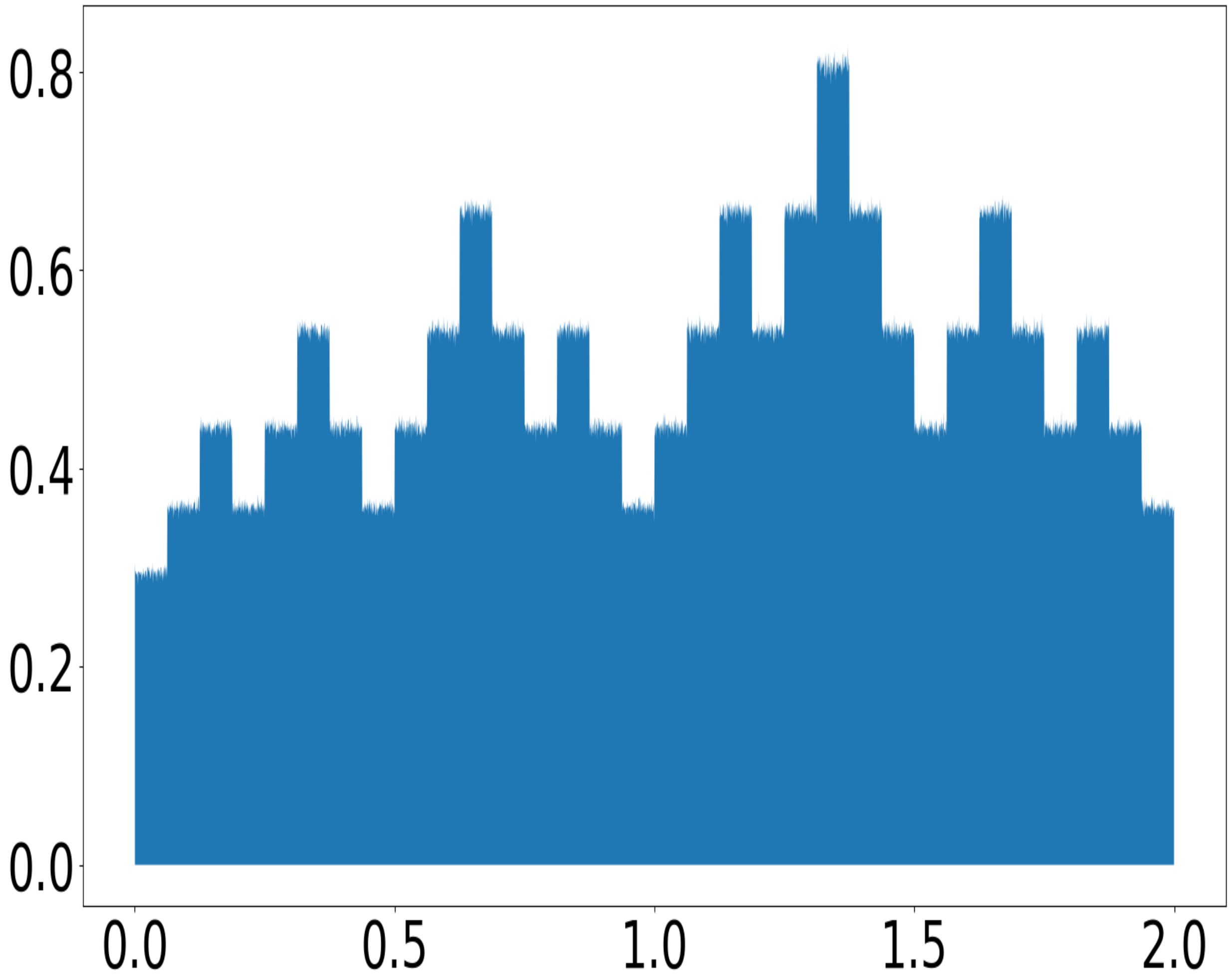




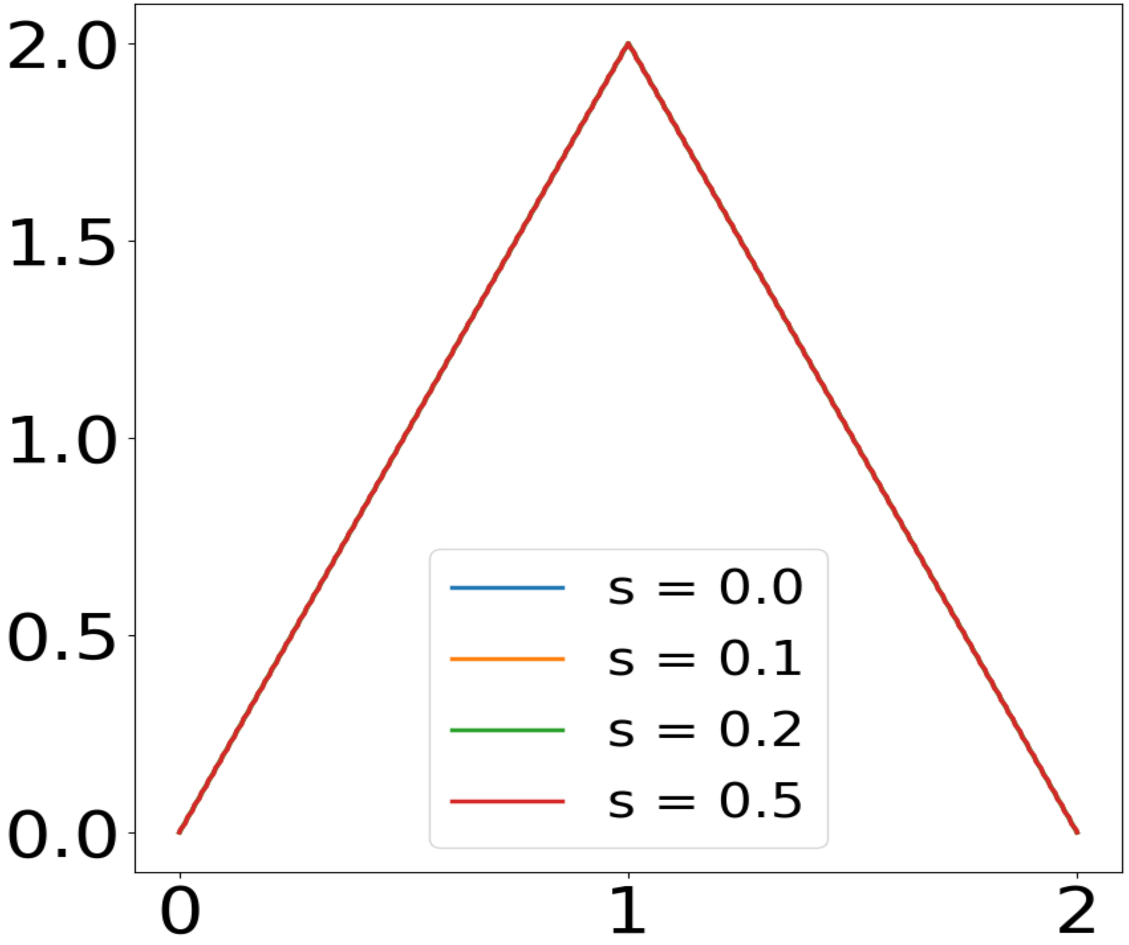
Can a butterfly control the climate?

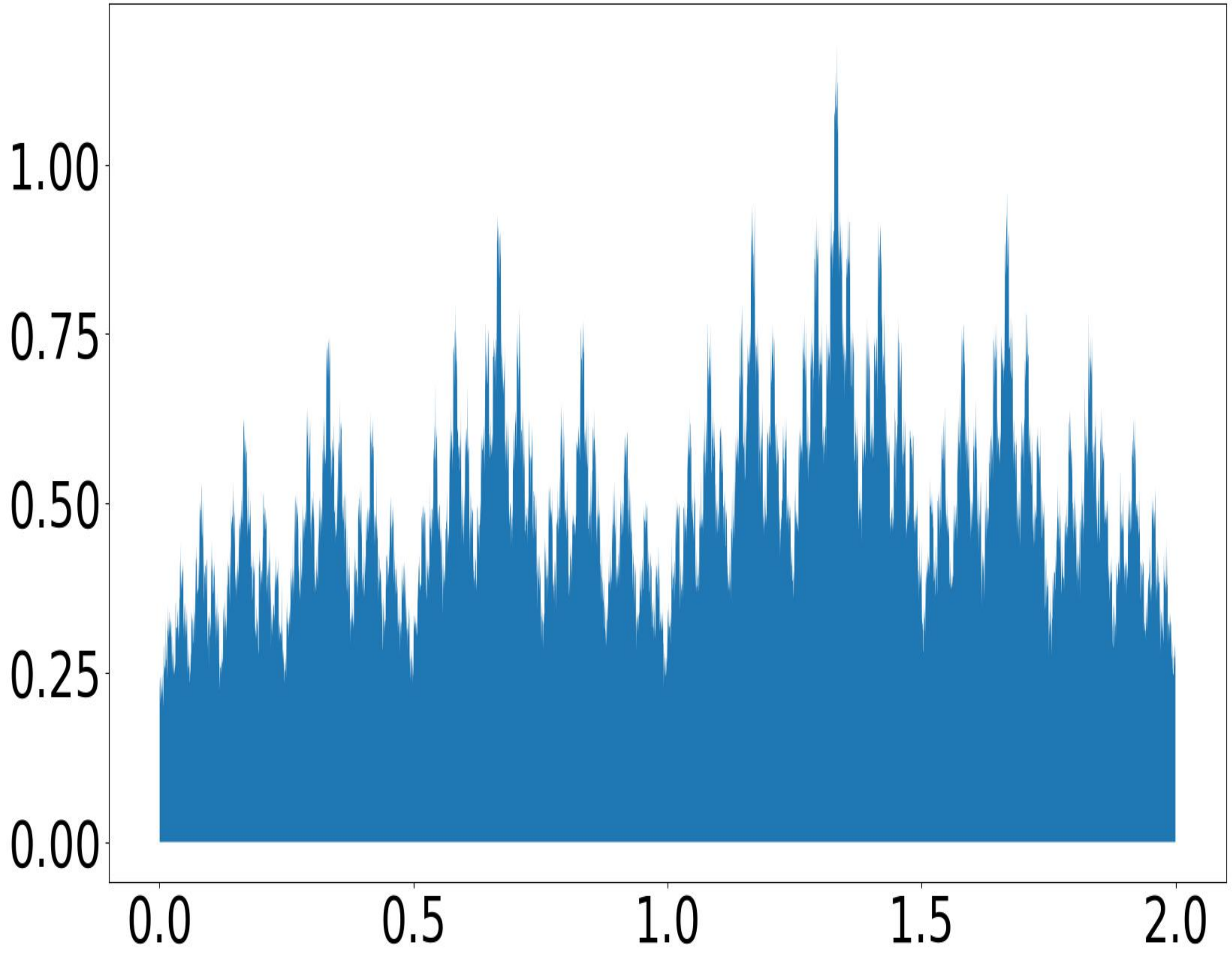
Evidence in model systems – smaller perturbation



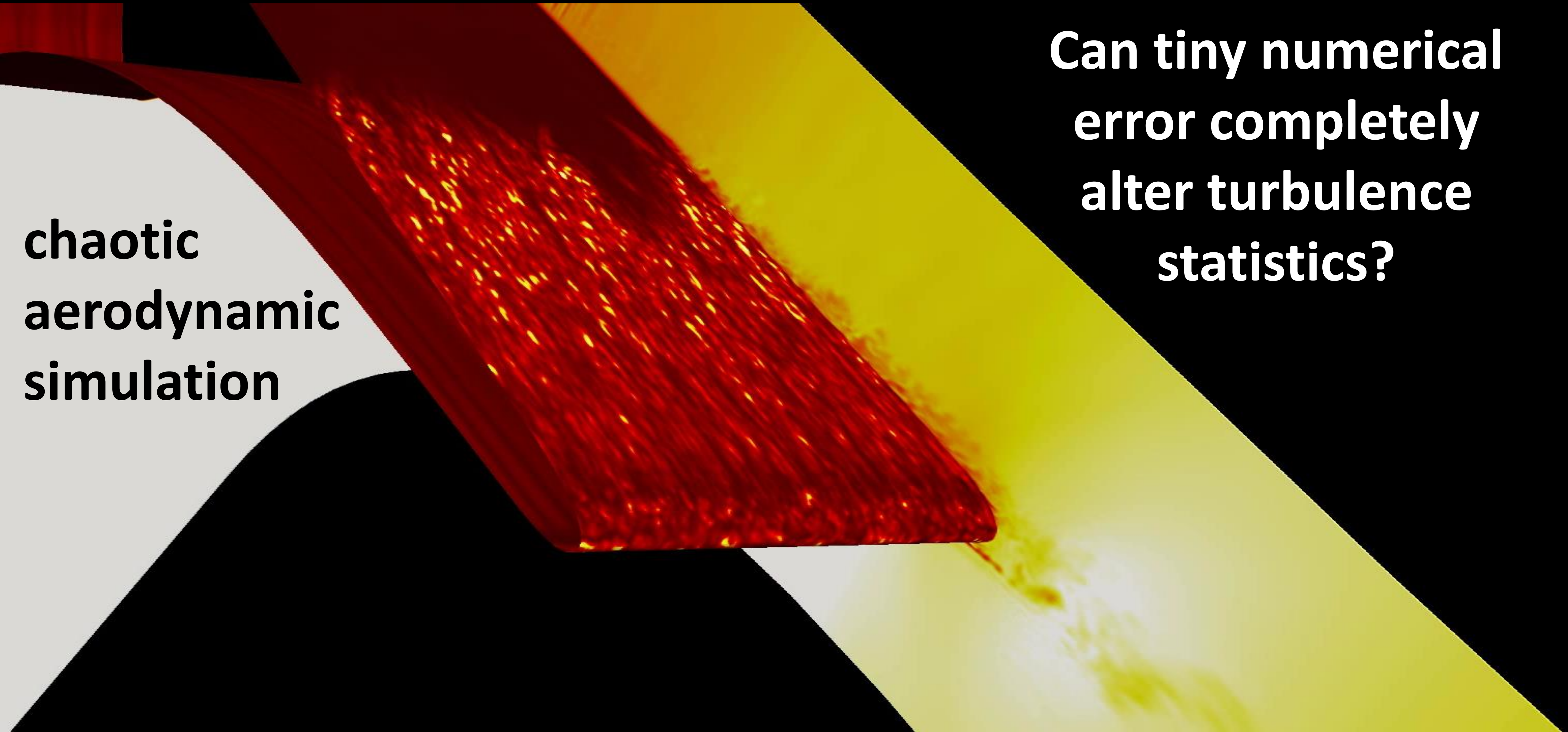


Can a butterfly control the climate?
Evidence in model systems – even smaller perturbation





What if a **butterfly** can **control the climate**?



Can tiny numerical error completely alter turbulence statistics?

chaotic aerodynamic simulation